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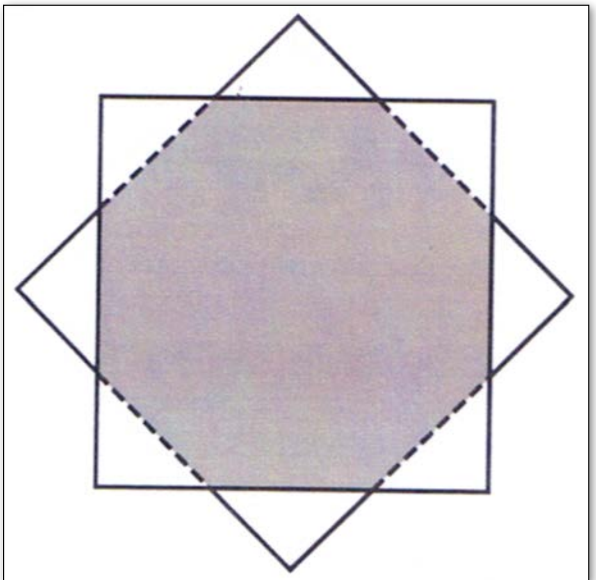
Zeno's Paradox - Some thoughts

Ancient Greek Philosopher Zeno of Elea (c.490 – c. 430 BC) is famous for his paradoxes, one of which, *Zeno's arrow paradox*, states that because an arrow in flight is not seen to move during any single instant, it cannot possibly be moving at all. (Not unless one understands the concepts of calculus, which came two millennia later!).

A related phenomenon, often called the Quantum Zeno Effect, was first discussed in John von Neumann's early work on the mathematical foundations of quantum mechanics [1]. In particular, in the rule sometimes called the "reduction postulate" he showed that, indeed, one can "freeze" the evolution of a system by measuring it frequently enough in its known initial state. Sometimes this is interpreted as saying that "a system can't change while you are observing it" or, more facetiously, "a watched kettle never boils".

The way this works is as follows (see the box for detailed derivation): the probability of a system, which is being measured with an apparatus described by a Hamiltonian H remaining unchanged (*i.e.* "surviving") while evolving by a short time Δt, is given by: $|\exp((-i/\hbar)H\Delta t)|^2 \approx [1 - [(1/\hbar)\,\Delta H\Delta t]^2$ provided that $[(i/\hbar)\,\Delta H\Delta t]$ is small enough.



▲ **FIG. 1:** Two sheets of Polaroid, the one behind turned by an angle θ (here shown at 45 degrees) allows the plane of polarisation of the transmitted light to be turned by the angle θ and the transmitted intensity be multiplied by $\cos^2\theta$, shown shaded.

Being quadratic in $[(1/\hbar)\,\Delta H\Delta t]$, this is simply equal to unity, meaning that if "observed" after a very short time, the system remains unchanged. Furthermore, if the observation is repeated at sufficiently small time intervals Δt, the system can be prevented from evolving at all. This Quantum Zeno Effect, or Quantum Zeno Paradox [2,3] is far from obvious but actually it has been verified experimentally [4] and is the subject of discussion in hundreds of articles [5].

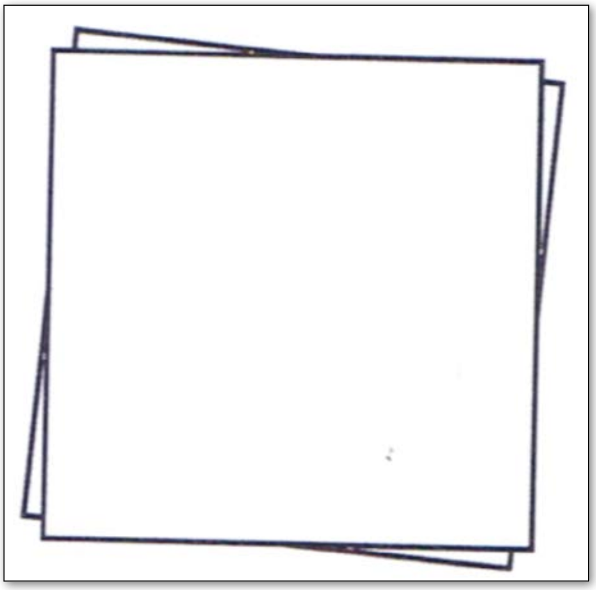
Quite a similar effect occurs in optics and may be demonstrated quite vividly, is a phenomenon that may also be termed a Zeno Effect. Here is how it works:

Consider a piece of polaroid in the shape of a square. Now imagine behind it an identical piece, rotated by an angle θ. (Fig.1, where θ = 45 degrees). The transmission of light going through both is proportional to $\cos^2\theta$ which, for small angles Δθ, is approximately equal to $(1 - \Delta\theta^2/2)^2$ *i.e.* negligibly different from unity, for sufficiently small Δθ, (*e.g.* 5 degrees, as in figure 2).

Now do it again with another sheet of polaroid, behind the first two, rotated by a further 5 degrees. Keep doing this 9 times for a total of 45 degrees. (Fig. 3). As you can see, the transmitted fraction of light through this stack remains near enough to one, *i.e.* zero absorption or zero change from the incident intensity - hence Zeno Effect.

However while absorption is to first order negligible, the angle of polarisation ends up being rotated by 5 degrees through every sheet, to a cumulative total of 45 degrees, in this example. So by being "measured" at sufficiently small intervals, the transmitted intensity is thereby being kept near enough equal to unity. However, the angle of polarisation can be rotated to any desired angle by a stack of a sufficiently large number of polarising sheets, gently rotated about the incident axis. A way of visualising this is by imagining a stack of Polaroids contained in a rectangular rubber tube, the back of which can be gently twisted with respect to the front so that the total angle is subdivided by N where N is the number of Polaroid sheets. (Circular Polaroids in a circular rubber tube would be easier to organise, of course, but would be harder to draw!).

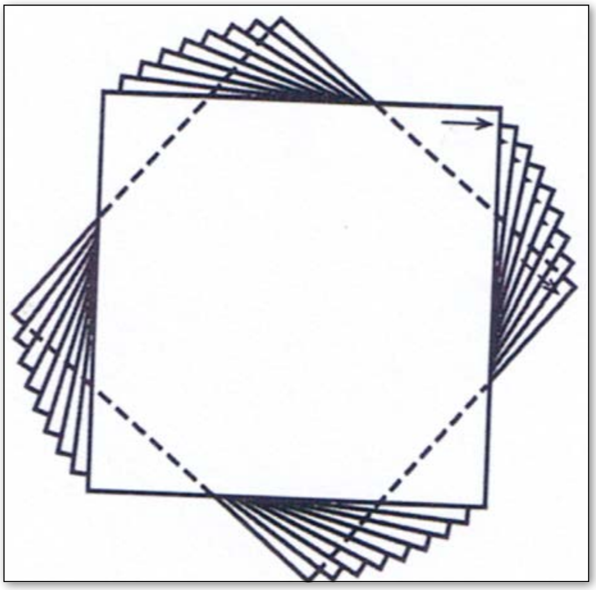
While this cumulative polarisation shift is essentially a classical effect, a quantum mechanical analogue has been shown [6] to lead to a geometrical phase shift *i.e.* a Berry



▲ **FIG. 2:** A piece of polaroid with a second piece placed behind at angle $\Delta\theta$ (5 degrees).

phase since that is a result of a first order variation of the parameter Δt. It is the second order variation $(\Delta t)^2$ that gives rise to the quantum mechanical Zeno Effect.

So, what about the original "arrow paradox" invented by Zeno? If we can think of the position of the arrow as due to first order changes of the time parameter, (the "instant") then the effect is clearly cumulative and results in the motion of the arrow, and hence there is no paradox. However, if we consider second-order changes of the "instant", interpreted as the velocity of the arrow, then of course, the velocity will stay constant! Is that what Zeno had in mind? But, of course, he didn't have calculus at his disposal!



▲ **FIG. 3:** Sheets of polaroid separated by 5 degrees. 9 sheets leads to a change of polarisation angle of 45 degrees, whilst the transmitted intensity is unchanged.

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Let us consider the state of the quantum system as $|\Psi_0\rangle$ at time $t = 0$ and the measurement is performed with an apparatus described by the Hamiltonian $\mathcal{H}$. The unitary time-evolution of the system due to the measurement is described by the operator $U(t) = \exp(-\frac{i}{\hbar}\mathcal{H}t)$, so that the state of the system at time t is given by $|\Psi(t)\rangle = U(t)\,|\Psi_0\rangle$. The probability that the system after the measurement will be in the initial state is: $P(t) = |\langle\Psi_0|\Psi(t)\rangle|^2 = |\langle\Psi_0|U(t)|\Psi_0\rangle|^2$. Expanding this for a small time-interval Δt gives:

$$P(\Delta t) = 1 - \left(\frac{\Delta t}{\hbar}\right)^2 (\Delta\mathcal{H})^2 + \ldots$$

with $(\Delta\mathcal{H})^2 = (\langle\Psi_0|\mathcal{H}^2|\Psi_0\rangle)^2$. Now performing N consecutive measurements in total time $t = N\Delta t$ this surviving probability is given by:

$$P^{(N)}(t) = [P(\Delta t)]^N = \left[1 - \left(\frac{t}{N\hbar}\right)^2 (\Delta\mathcal{H})^2\right]^N + \ldots$$

which goes to 1 for large N.