

# NUCLEI

## “AT THE TOP OF THEIR SHAPE”

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*The atomic nucleus is an ideal laboratory not only for physicists, but especially for mathematicians. Indeed, nowadays a wide variety of theories based on quantum mechanics as well as semi-classical approaches have been applied to nuclear structure and reactions and innumerable experiments have been performed to test them. The most striking feature of the nucleus is that it continues to challenge physicists of all sorts.*

This is because it displays an everyday richer phenomenology that underlies abstract mathematical concepts, such as group theory and Lie algebras, which are used to describe “hidden” symmetries. Those symmetries, contrariwise to more commonly used ones, such as translational invariance, parity, time-reversal, or even geometric symmetries, are not properties of the object itself, but rather are contained in their quantum mechanical Hamiltonian. We will dwell, in the following, on the Bohr Hamiltonian and the newly discovered symmetries that are associated with it.

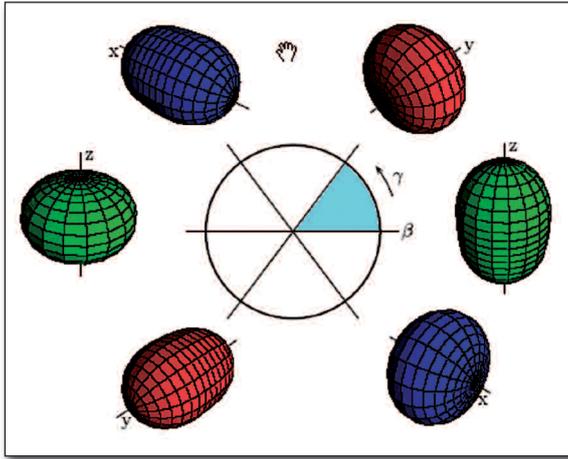
### Droplets and quadrupoles

Since the beginning of nuclear physics the phenomenology of the atomic nucleus has been profitably described in terms of a droplet of a quantum liquid. This semi-empirical model, due to von Weizsäcker [1], incorporates a number of terms that comes from different physical considerations into a mass-formula. This approach was of a static nature, but the idea of a liquid was the starting point of a more elaborate quantum description that takes in the dynamics of surface oscillations of the drop. The celebrated model of Bohr and Mottelson [2], that dates back to the fifties, describes certain properties

of the nucleus in terms of its surface that, under the influence of a restoring force, is allowed to perform quadrupole (or, more generally, multipole) oscillations around an equilibrium ellipsoidal shape that can be spherical or deformed. Each shape is uniquely defined by five variables: two,  $\beta$  and  $\gamma$ , that encode the extent of quadrupole deformation and the asymmetry in the intrinsic frame of reference and three Euler angles, that govern the orientation in space of the intrinsic ellipsoid with respect to a laboratory frame. The first parameter,  $\beta$ , may be thought of as the radius in a suitable two-dimensional polar frame of reference, while the second,  $\gamma$ , is an angle that may range from 0 to  $2\pi$ . Each point in this two-dimensional plane is uniquely associated with a given ellipsoidal shape in the intrinsic frame of reference. The three Euler angles that specify this orientation do not give further information on the intrinsic shape and therefore one can then confine this two-dimensional plane to a  $60^\circ$  wedge, because all the other wedges may be obtained by simply re-labelling the axes of the intrinsic frame (see Fig.1). The Bohr Hamiltonian, which is written in the intrinsic frame of reference as a partial differential equation in those five variables, is the quantized Hamiltonian for the

▲ Three-dimensional representation of a nucleon: quark and gluon structure of the proton. © Photothèque CNRS / CNET / Lactamme / J.-F. Colonna / URA756

► **FIG. 1:** Polar plane for the description of quadrupole shapes as a function of the radial variable  $\beta$ , the quadrupole deformation, and the angle  $\gamma$ , the degree of asymmetry (the so-called Hill-Wheeler coordinates). The fundamental  $60^\circ$  wedge (enlarged in fig. 2) is sufficient to describe all possible intrinsic ellipsoidal shapes [2,3] because any other point in the diagram can be obtained by simply relabelling the intrinsic axes.



quadrupole oscillations of an ellipsoidal surface. It contains three kinetic terms (in  $\beta$ , in  $\gamma$  and rotational) as well as a potential term,  $V(\beta, \gamma)$ . This last term, which is a function of  $\beta$  and  $\gamma$ , is particularly relevant to the present discussion, because it represents the potential associated with some sort of restoring force on the nuclear surface, that may be thought of as the response of the surface to an external stimulus. In principle this potential is not known and the original solution of Bohr and Mottelson used a  $\gamma$ -independent harmonic oscillator in  $\beta$  centered around the origin with the idea of describing the oscillations of a spherical surface [2].

**Ellipsoids**

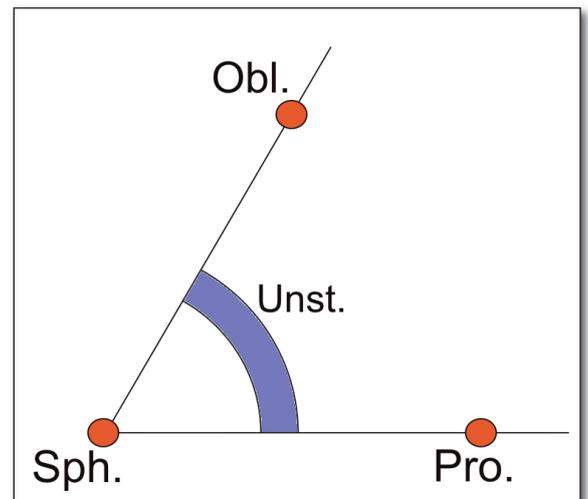
Since that successful solution a number of important analytical, approximated and numerical solutions of this Hamiltonian has been found (they are collected in Ref. [3]) and their applications to nuclear spectroscopy at low-energy have been very rewarding.

In particular three classes of solutions have been traditionally discussed which correspond to different shapes: the sphere, the prolate and oblate axially deformed ellipsoid, and the so-called  $\gamma$ -unstable [4]. Each of these classes comes from a potential surface with a particular position of the minima in the  $\beta$ - $\gamma$  plane as illustrated in Fig. 2: i) when the absolute minimum is at zero the surface is spherical, ii) when it is a point with  $\beta \neq 0$  along the  $\gamma=0$  (or  $\gamma=\pi/3$ ) axis the surface is a prolate (or oblate) axial ellipsoid and iii) when the locus of minima is a sort of circular valley one speaks of a  $\gamma$ -unstable shape [4]. In addition when the minimum is a point within the wedge, the surface is a triaxial ellipsoid. During the second half of the last century until the present time, each given shape has been the subject of many investigations which from one side have pushed the technical ability of nuclear spectroscopists (high-efficiency multi-detector arrays have made it possible to look for fine structures in spectra) and from the other have made

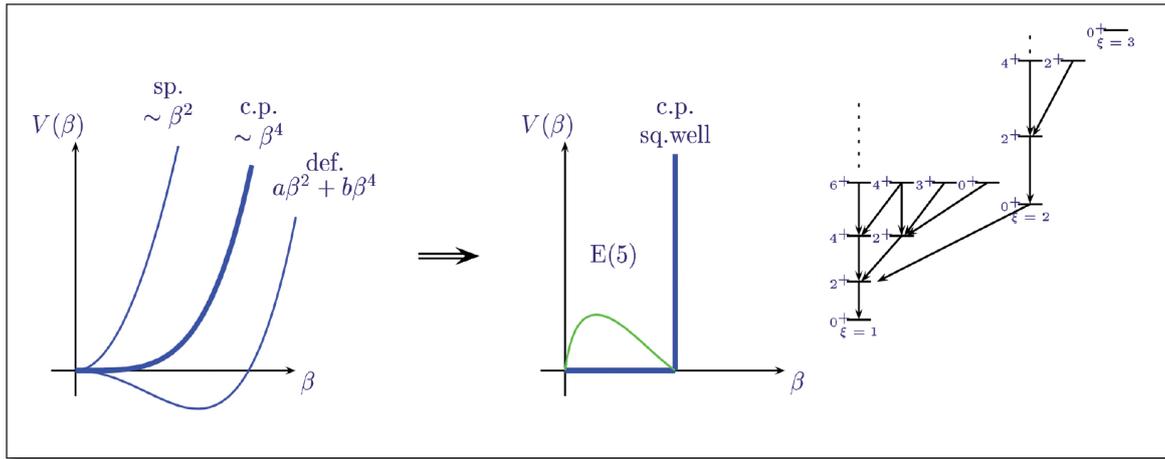
richer our exploration of the nucleus. Just to mention a few interesting phenomena: the super- and hyper-deformation, which might occur at high spin and temperature, and their relation to fission reaction and clustering, wobbling motion, search for triaxiality and super-deformed triaxial shapes, band termination, prolate-oblate shape coexistence, octupolar and exotic shapes (See, for instance, [1,14,15,16]). Each of these topics (a few of which are well-covered in Ref.[16]) would require a separate article, but this is beyond the scope of the present paper.

**Interacting Boson Model**

These three classes of potential surfaces with different minima have been related to the group structures,  $U(5)$ ,  $SU(3)$  and  $SO(6)$  for spherical, axially deformed and  $\gamma$ -unstable shapes respectively. This has been possible especially after the inception of the Interacting Boson Model [1,5], a very successful algebraic approach proposed by A.Arima and F.Iachello in the mid-seventies, that parallels the collective description. In the IBM the nucleons in an even-even isotope are divided into an inert core and an even number of valence particles. These particles are then considered as coupled into two kinds of bosons (an effect essentially due to the combination of pairing and quadrupole-quadrupole interactions) that may carry either a total angular momentum 0 or 2, and are respectively called the s- and d-bosons. The bilinear operator that may be formed with the s- and d-boson creation and annihilation operators close into the  $U(6)$  algebra, whose three possible



▲ **FIG. 2:** On the fundamental wedge, the spherical limit correspond to the  $\beta=0$  point, axially deformed prolate and oblate limits correspond to a point with non-null  $\beta$  and  $\gamma=0$  and  $\pi/3$  respectively. The  $\gamma$ -unstable limit corresponds to the locus of points with non-null  $\beta$  and  $0 < \gamma < \pi/3$ . Isolated minima in all other points (not shown) represent triaxial shapes.



◀ FIG. 3: The potential  $V(\beta)=a\beta^2+b\beta^4$  (left) spans from spherical to  $\gamma$ -unstable deformed cases, crossing the critical point (c.p.). In the E(5) solution (centre) the  $\beta^4$  potential at the critical point is approximated by an infinite square well. The ground state radial wave function, which is essentially a Bessel J function is shown in green. The lower portion of the analytic spectrum of the E(5) critical point symmetry is shown on the right. The slightly anharmonic eigenstates are labelled by  $J\pi$  and by a family index  $\xi$ . Arrows indicate electric quadrupole transitions.

subgroup chain match the U(5), SU(3) and SO(6) solution of the Bohr Hamiltonian. The fact that the L=2 has a privileged role in this model explains why the quadrupole degree of freedom emerges naturally from the abstract algebra. On the other hand the necessity of including L=0 monopole bosons focuses attention on the short-range pairing interaction.

A rationale for the valence bosons ansatz is the simplicity of the model: in fact the IBM allows for treatment of heavy nuclei, where sometimes the more natural shell-model approach would need a much larger set of basis states. Although the IBM is a phenomenological model, the connection with microscopic aspects is ensured by the N-dependence across a given shell.

With the clear-cut algebraic formulation of the IBM, people have investigated its relations to the collective model and have transposed the group labels to the collective description using them as benchmarks for nuclear structure. Although a precise mapping from one model to the other is far from being trivial [17], it is clear that the collective model and the interacting boson model describe the same phenomenology with a geometrical and with an algebraic approach respectively. Roughly speaking the Bohr-Mottelson collective description can be recovered by taking the large N (boson number) limit of the IBM.

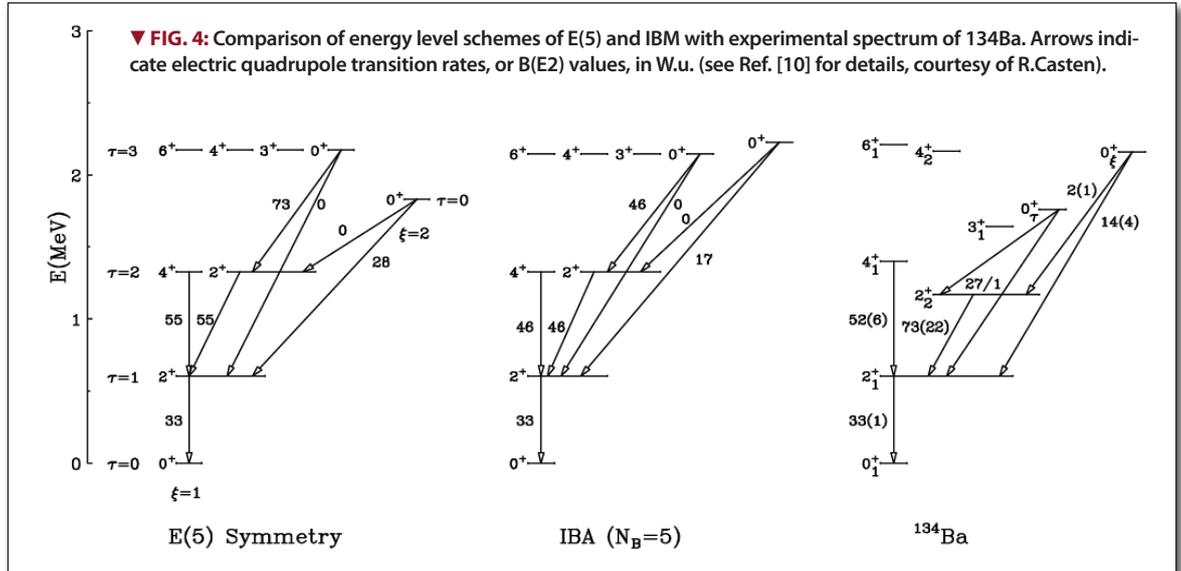
### Shape-phase transitions

As has been said, a number of exact solutions of distinctive importance may be given, but alongside the three traditional ones, there are a few which have come to the fore recently because of the novelties they imply (see [3] and [13]). In fact, after one associates a certain phase to each of the shapes described above, the problem of studying the so-called shape-phase transitions between the various classes of solutions of the Bohr-Mottelson model comes forward. For example a  $\gamma$ -independent potential of the type  $V(\beta)=a\beta^2 + b\beta^4$  is

able to generate either a spherical minimum, or a deformed  $\gamma$ -unstable one, depending on the values of the parameters a and b (see Fig. 3). Of course this potential allows one to span the whole range of intermediate cases, including the critical point. In this case one speaks of a U(5)-SO(6) quantum phase transition or a spherical to  $\gamma$ -unstable transition (which is a second-order phase transition in Ehrenfest's classification [6,7]). The critical point, that from the theory of phase transitions is defined as the value of the control parameter for which a given order parameter (or one of its derivatives) manifests a discontinuity, is obtained in the case of a pure  $\beta^4$  potential. This potential is not soluble analytically, but numerical solutions have been found [8]. Before the numerical solution Iachello (in a series of papers that cover this and other cases) has realized that the pure quartic potential may be approximated with an infinite square well in  $\beta$ , that yields an exactly solvable problem [9]. The analytic solution that he has found in this way was named E(5), because the eigenfunctions, that are essentially spherical Bessel functions, form a basis for the Euclidean group in five dimensions. Another reason to do so is that E(5) is indeed a spectrum generating algebra (In other words the Hamiltonian is a polynomial in the generators of a Lie algebra and this implies an easy calculation of the matrix elements and therefore an easy diagonalization.)

for the Bohr Hamiltonian with the infinite square well potential: where the potential is null, the whole equation reduces to just  $\pi^2$ , the five-dimensional vector momentum squared, or, in other words, the generator of translations and rotations in five dimensions, that is the Euclidean group.

**Nuclear phases coming from the quadrupole degree of freedom have been pictured in a phase diagram, called the Casten triangle**



The E(5) solution is not just a nice mathematical solution of the Bohr Hamiltonian at the critical point of the shape-phase transition between spherical and deformed  $\gamma$ -unstable regimes, but serves as a paradigm for nuclear structure, because it is parameter free (except for an overall energy scale) and gives precise formulas for energy eigenvalues, wave functions, electromagnetic transition rates and selection rules that may be tested against spectroscopic data.

Although, with respect to the collective solution, a correction for the finite number of particles is due (and may be easily done within the IBM), various nuclei have been identified as candidates for this critical point symmetry, as it is now dubbed, notably  $^{134}\text{Ba}$  [10] and other isotopes [11]. In Fig. 4 the E(5) spectrum is shown on the left, while the spectrum obtained in the Interacting

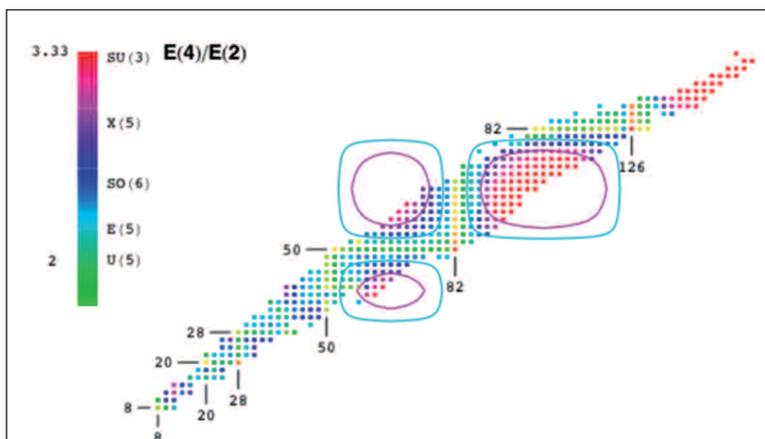
Boson Approximation considering  $N=5$  bosons is shown in the centre. On the right we have the comparison with the measured spectrum of  $^{134}\text{Ba}$ . The symmetry, although slightly broken, can be recognized not only from the relative energy of the various states, but also from the calculation of quadrupole electromagnetic transitions between them, indicated with arrows and other measurable properties such as isomer shifts, transfer intensities and so on (see Ref. [10] for a more complete description of the quantum numbers and notation).

The follow up of the E(5) work has been a similar analysis [12], in which an approximate solution for the critical point between the spherical and axially deformed shapes has been proposed and put in correspondence with some underlying symmetry of an unspecified nature, named X(5).

In this case the potential in  $\beta$  is again an infinite square well, but in order to obtain a minimum around  $\gamma=0^\circ$  a harmonic oscillator is used for the potential in the  $\gamma$  variable. Although a thorough description of this solution would lead us astray from the intended goal of this paper, it must be said that it has been very fruitful in terms of comparison with experiments and the identification of candidates has been quite successful, because spherical nuclei are abundant along closed shells, while axially deformed ones are abundant at mid-shell. Because of the many isotopes that sit in the intermediate region (see Fig. 5), encountering a transitional nucleus there becomes a very likely event.

### Signature

The easiest experimental signature for each type of behaviour is the ratio of the energy of the first  $J\pi=4+$  states over the energy of the first  $2+$  state. This is not alone sufficient to ensure that an isotope belongs to a particular class or phase, but is a first pointer to classifying



the spectra and the nature of the nuclear shape. This ratio is plotted in Fig. 5 for a large portion of the nuclear chart as a function of the number of neutrons and protons (in abscissa and ordinate respectively). The ratio for the U(5) case is 2 since the spectrum is harmonic, and examples are found along the lines of magic numbers (closed shells). The ratio for the E(5) case is  $\sim 2.09$  and corresponds to the sky-blue colour, while for X(5) is around 2.9 and corresponds to violet. The SO(6) limit has a ratio of 2.5, while the pure rotational phases have a ratio of  $10/3 \approx 3.33$  which is found at the very centre of a shell for both protons and neutrons. Other more refined quantities, such as electromagnetic transition rates, isotope and isomer shifts and even pair-transfer intensities may be used to pin down the critical point and the onset of shape phase transitions.

The nuclear phases coming from the quadrupole degree of freedom (either in the collective Bohr-Mottelson description, or in the algebraic IBM) have been pictured in a phase diagram, called the Casten triangle, that has been subsequently enlarged to the extended Casten triangle (see Fig. 6). At the three vertices there are the spherical phase, characterized by U(5), and axially deformed phases. These last two share the same algebra, SU(3), but describe either the prolate (rugby ball) shape or the oblate (mandarin orange) shape. The  $\gamma$ -unstable phase sits on the side between those two, while the X(5) and E(5) critical points are intermediate between the various phases. Although most nuclei cannot be exactly put in correspondence with the special points of the triangle, it is often possible to use those points as benchmarks.

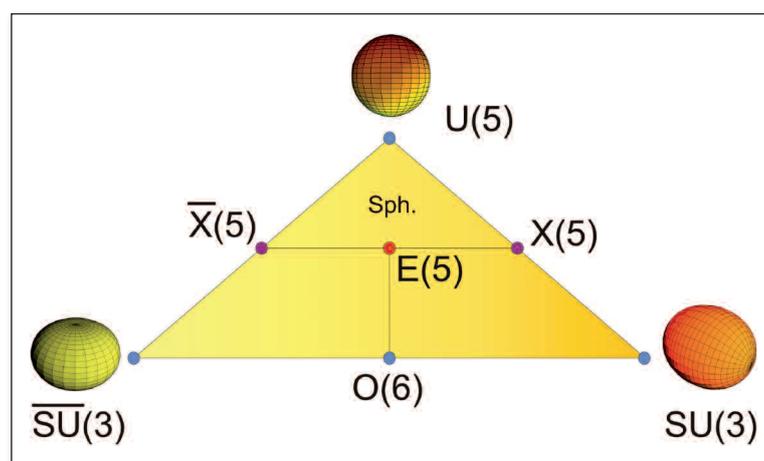
Since the year 2000, these studies have spurred a large number of experiments dealing with a precise gamma-ray nuclear spectroscopy that have been performed or are currently being carried out in major laboratories in Europe and in the rest of the world and they have also promoted a great deal of theoretical interest aimed at better understanding the shape-phase transitions and their underlying symmetries (see e.g. [12]). The main conclusion, that can be drawn from the interplay between the theory of Lie algebras and the exact solutions of simple models of nuclear structure, is twofold: from one side, in a bottom to top fashion, it furnishes a very precise way to classify and give a proper name to the extraordinary variety of observations that have been carried out, but more importantly, in a top to bottom perspective, new mathematical solutions that are inherent to these algebraic models might fuel new waves of experimental campaigns and give us a better understanding of the atomic nucleus.

The overall picture of the quadrupole nuclear collective behaviour, that emerges from the works summarized in the present article, is fairly complete and strongly

supported by experimental evidence and gives us a detailed understanding of the nuclear shape and the fascinating mathematics underlying it.

### About the author

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▲ FIG. 6: Extended Casten triangle. Spherical and axially deformed phases are indicated at the three corners. The  $\gamma$ -unstable one is on the base while the three critical points are given in violet (first order phase transitions) and red (second order).