Nonextensive statistical mechanics : implications to quantum information

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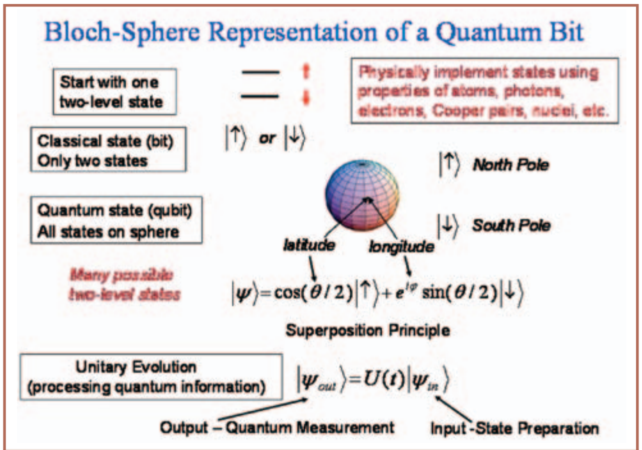
The interactions and correlations among the constituents of many-body systems are manifested in characteristic physical properties such as ferromagnetism, superconductivity, etc. A list of some that have been studied in the last century is given in the Box (see below). A parallel development in quantum information was initially slow but over the past two decades progress has been very rapid. Fundamentally this is another aspect of quantum correlations in composite systems arising from the twin features of the superposition principle and the tensor product structure of state space. These features are not utilized in the same manner in quantum many-body physics. In the Box, a corresponding parallel list with properties of many-body systems is given because there has been an interplay between the two research efforts and their understanding. The basic quantum mechanical principles apply to both cases except different aspects are utilized because the goals are different in each.

Both areas of investigation are based on a probabilistic foundation with a variational underpinning founded on an "entropy" maximization, which may be called Quantum Statistical Mechanics [1]. The specific form of the entropy as a functional of the

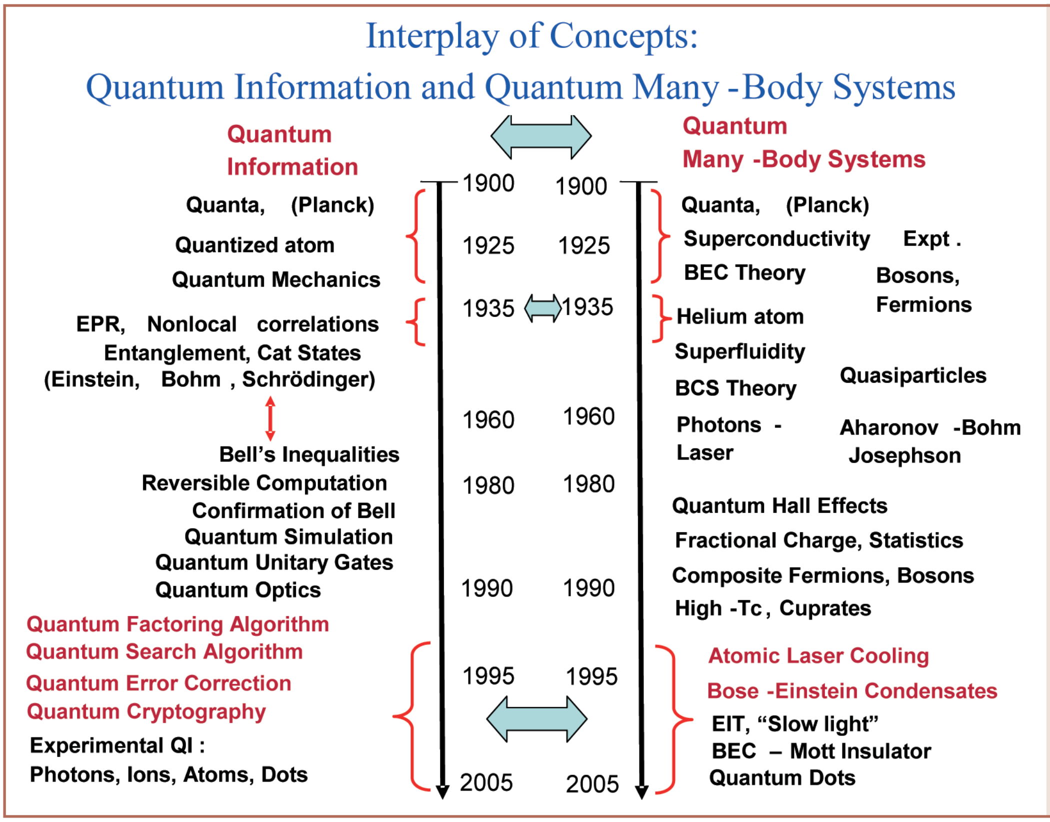


▲ **Fig.1:** Properties and Features of a Single Quantum Bit

density matrix will be made explicit presently. See Table I for definitions of density matrix and associated quantities.

The equilibrium properties of the many-body systems are then given by a maximization of von Neumann entropy subject to certain constraints such as the average value of the Hamiltonian of the system - "energy". This leads to the familiar exponential probabilities of the Boltzmann-Gibbs (BG) form. Any non-equilibrium properties are studied by a quantum time evolution equation. Non-equilibrium properties such as the anomalous relaxation in time are often analyzed with the quantum version of the Tsallis entropy with an entropic index, q, which leads to power- law probabilities in contrast to the BG type [2]. (See the Box in the Introduction of Tsallis and Boon).

In quantum information, the maximum entropy scheme is not of use because its origins are elsewhere as will be made clear presently. The evolution is replaced by processing of information



◄ **Box:** Explanation of the terms

EPR – Einstein, Podolsky, Rosen;
QI – Quantum Information;
BEC – Bose – Einstein Condensation;
BCS – Bardeen, Cooper, Schrieffer;
EIT – Electromagnetically Induced Transparency

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accomplished by quantum operators, such as unitary or measurement operators, enabling the passage of given initial information to a known destination. The energy level (band-) structure of the states plays the central role in determining the global physical characteristics of the many body system which led to the Silicon-based classical computer. In this development, the classical information structure was sufficient in its construction and operation. In contrast, the quantum information structure exploits the state function of the many-body system and the corresponding development of a quantum computer is presently at an infant stage and it will perhaps be based on quantum optics or state-space aspects of condensed matter. The Box gives a flavor of this mutual relation between many-body systems and information theory, highlighting the twin aspects of the quantum energy level structure and the quantum state.

Classical information theory is a description of communicating information (signals, alphabets, etc. denoted by real numbers x_i, i=1, 2,…N) from one place to another using systems governed by classical physics. It rests on four intuitive ideas: (1) probability structure of a message source, $p(x_i)$; (2) a notion of additive information content, $I(x_i)= -\log p(x_i)$, where it is the sum of each if there are independent sources; (3) use of a binary arithmetic (see Fig. 1 - yes, no - classical bit), which is sufficient to develop coding, error correction, etc. associated with transmission and reception of information, and a unit of information, the classical bit, log2; and (4) an additive measure of information, which quantifies average information per symbol, the von Neumann entropy, $S = -\sum_{i=1}^{N} p(x_i) \log p(x_i)$. When more than one source is considered, generalization of these concepts lead to the notions of (a) marginal probabilities, (b) conditional probabilities, and related entropies, and (c) relative entropy which enables comparison of two sources. The above description is for digital sources. There are also continuous sources (e.g. light) which describe analog systems. All this changes dramatically when quantum physics is the underpinning structure. To appreciate this change, we first display the foundations of quantum theory that subsumes classical theory (see Table 1 for the relevant definitions).

Quantum theory involves (a) superposition principle, (b) uncertainty principle, and (c) the system density matrix governing the probabilistic description of the system. Physical quantities associated with the system are represented by Hermitian operators whose average values defined in terms of the system density matrix give their measured values. The density matrix replaces the probability of occurrence of events in classical theory. The classical bit now takes on a more general representation, because in the quantum description the superposition principle comes into play. See Fig. 1, for a pictorial representation of a qubit. Conceptually these three features give a more general description of the system than the classical theory. Thus the classical information theory based on probabilities associated with signals and the consequent theoretical structure defining entropy as the information measure, algorithms, coding of information, etc. are all generalized in the quantum version with important consequences. The superposition principle precludes cloning and deletion of information and significantly improves the classical search algorithm. The uncertainty principle places conditions on measurements of a certain class of physical variables of the system and when there is more than one signal or source, gives conditions for "independence" or "separability" of the systems. In fact, classically correlated signals become generalized to include entanglement and other nonlocal features in the quantum context. See Fig. 2 for a description of these concepts in the simple case of two qubits. More precisely, quantum entanglement implies that the parts do not determine the whole. This feature gave rise to dense coding, "teleportation" and novel quantum cryptography that have no classical counterparts. Grover's quantum search algorithm exploits the superposition principle and makes the search faster than the classical version.

A.

Density matrix: $\rho = \sum_i q_i |i\rangle\langle i|; \{q_i\}$, **Probabilities:** $0 \le q_i \le 1$, $Tr\rho \equiv \sum_i q_i = 1$

B.

In general $\rho^2 \le \rho$

Pure state: $\rho^2 = \rho \Rightarrow \rho_P = |\Psi\rangle\langle\Psi|$, $|\Psi\rangle \equiv$ **Pure state of system**

Mixed state: $\rho^2 < \rho$; **Representation as in A.**

C.

Composite system density matrix: $\rho(A,B)$

Marginal density matrices are:

$$Tr_B \rho(A,B) = \rho_1(A),; \quad Tr_A \rho(A,B) = \rho_2(B)$$

D.

Tsallis entropy: $S_q = (q-1)^{-1} Tr\{\rho - \rho^q\}$

von Neumann entropy is obtained when q=1:

$$S(A,B) = -Tr_{A,B}\rho(A,B)\ln\rho(A,B) \ge 0$$

E.

If composite system is uncorrelated i.e., $\rho(A,B) = \rho(A) \otimes \rho(B)$ **then**

$S_q(A,B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$ **(NONADDITIVE PROPERTY)**

For q=1 this gives ADDITIVE PROPERTY for the von Neumann entropy.

F.

Comparison of two systems : **Fidelity:** $F = Tr(\rho(A)\rho(B))$

Relative entropy:

$$K(A|B) = Tr\rho(A)[\ln\rho(A) - \ln\rho(B)] \ge 0$$

G.

For a pure state of composite system, $|\Psi(A,B)\rangle$, **marginal density matrices are**

$\rho(A) = Tr_B |\Psi(A,B)\rangle\langle\Psi(A,B)|$, $\rho(B) = Tr_A |\Psi(A,B)\rangle\langle\Psi(A,B)|$ **and if this system is entangled, then the entanglement of formation is given by**

$$S(A) = -Tr\rho(A)\ln\rho(A).$$

▲ **Table 1:** Quantum Density Matrix Description

The Shor quantum factorization algorithm is another successful application. For a detailed exposition, one may consult [3].

The hallmark of quantum physics is quantum non- locality which often involves entanglement, whereby distant systems can exhibit random yet perfectly correlated behavior. A fundamental problem in quantum information science is the characterization of entangled states and their measurement. Originally this was stated in terms of a violation of a certain inequality due to Bell by measuring a sequence of correlations that could not be explained by any local realistic model. This violation was taken to indicate quantum non- locality. Experimental tests of the Bell inequality have established this feature of quantum theory. While this violation detects entanglement, it does not quantify it nor is it guaranteed to succeed. Werner defined the separability of states as follows: if the system density matrix, $\rho(A,B)$, of a composite system (AB) (see Table 1) can be written as a sum of the products of the density matrices of its components, $\rho_i(A)$ & $\rho_i(B)$ in the form $\rho(A,B) = \sum_i w_i \rho_i(A) \otimes \rho_i(B)$, $0 \le w_i \le 1$, and $\sum_i w_i = 1$, then the system is separable Those composite systems for which this decomposition does not hold are said to be entangled. A criterion for testing this property was first stated by Peres: if the density matrix of the composite state does not retain its property of positive semi-definiteness under the action of time reversal of one of the subsystems, then the system is entangled. The entanglement measure is often stated in terms of "entropy of formation", originally formulated in terms of the von Neumann entropy for pure states given in Table 1. Since entanglement is due to intrinsic correlations among the parts making up a system, it is not obvious that one could employ an additive measure for this purpose. This point is the subject of discussion to this day [4]. An additive measure of a system is defined by the sum of the corresponding measures of its components (see Table 1). If the system is entangled, it is not clear that additive measures such as von Neumann entropy would be appropriate in general. Anticipating the possibility of a non-additive feature of entanglement, the Tsallis entropy was employed to characterize it. This was shown to be more successful in correctly obtaining the separability criterion of a certain known state where the von Neumann measure gives the wrong answer [5, 6, 7, 8, 9]. Table 2 gives a summary of these results. In this Table, we consider a simple special composite mixed state of two qubits called the Werner state, which is a sum of pure state density matrix and a density matrix representing noise: $(I_2 \otimes I_2)/4$. By noise is meant that all the four states of this system occur with equal probability, 1/4. This state has the interesting property of being an entangled pure state for F=1, but a separable mixed state for certain values of F. The separability condition is deduced by various methods and these are compared with the exact result obtained by the Peres criterion in Table 2.

The Werner state [8] is a mixed state and is given by

$$\rho_W(A,B) = F\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + \frac{(1-F)}{3}\left(I_2 \otimes I_2 - \left|\Psi^-\right\rangle\left\langle\Psi^-\right|\right)$$

Here F is a parameter in the range (1/4, 1) and $\left|\Psi^-\right\rangle = \left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)/\sqrt{2}$

Separability conditions for this state for values of F is determined by various methods:

(a) Peres - Horodecki partial transpose: $F \le 0.5$

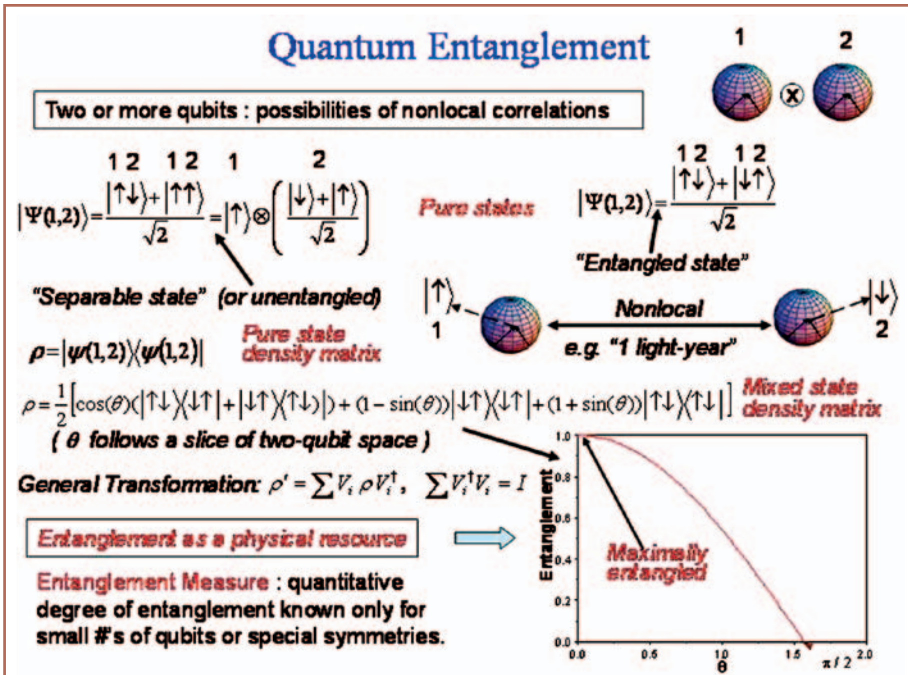
(b) von Neumann conditional entropy(q=1): $F \le 0.807$

(c) Bell inequality: $F \le 0.78$

(d) Tsallis conditional entropy: $F \le 0.5$
$\left(\lim_{q\to\infty} S_q(A|B) \ge 0\right)$

This has been extended to a general 3-parameter Werner state by Tsallis et al [8] and to the N-dimensional Werner qubit state by Abe [8]. The Peres-Horodecki condition (a) is known to be exact for two qubits. Thus the non-additive Tsallis approach (d) is found to be better than the additive von Neumann scheme (b) in all forms of the Werner state considered [8].

▲ Table 2: Comparison of Separability Criteria of the Werner State



▲ Fig.2: Illustration of quantum entanglement based on a single two quantum-bit example. Salient features of pure and mixed state density matrices are also illustrated.

Another fundamental issue is the proper discriminating measure when two systems are under consideration. In classical information theory, one employs the Kullback – Leibler relative entropy for this purpose which also has its quantum version. These are also additive measures and the Tsallis counterparts of these have been put forward and employed in the quantum context as well [10, 11]. There is promise in future work using the Tsallis approach to problems arising in quantum information theory, especially in the areas of quantum algorithms and quantum computing.

There has been some discussion of the thermodynamics of information, in particular quantum information. Since there are hints that quantum entanglement may not be additive, and since the concept of entropy has been introduced into the discussion, an examination of maximum Tsallis entropy subject to constraints such as the Bell-Clauser- Horne-Shimony-Holt observable was studied for purposes of inferring quantum entanglement [5, 6].

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Entropy and statistical complexity in brain activity

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Electroencephalograms (EEG) are brain-signals that provide us with information about the mean brain electrical activity, as measured at different sites of the head. EEGs not only provide insight concerning important characteristics of the brain activity but also yield clues regarding the underlying associated neural dynamics. The processing of information by the brain is reflected in dynamical changes in this electrical activity. The ensuing activity-variations are found in (*i*) time, (*ii*) frequency, and (*iii*) space. It is then very important to have theoretical tools able to describe qualitative and quantitative variations of these brain-signals in both time and frequency. Epileptic signals (ES) are specially important sources of brain information. We concentrate on ES in this article. The EEG-signal is what mathematicians call a non stationary time-series (ST). Powerful analytical methods have been developed over the years to extract information from ST. The brain ST is contaminated by another body-signals (called artifacts) due mainly to eye movements and muscle activity. Artifacts related to muscle contractions are specially troublesome in the case of epileptic seizures that exhibit rigidity and convulsions (called tonic-clonic seizures). The troublesome artifacts acquire here very high amplitudes that contaminate the whole recording. A drastic way of preventing this contamination is by injecting curare to the patient. The classic work of this type is that of Gastaut and Boughton (GAB) [1], who described the characteristic frequency pattern of a tonic-clonic epileptic seizure (ACES) in patients subjected to muscle relaxation from curarization and artificial respiration. They found that, after a short period characterized by phase desynchronization of the brain' signals, a typical feature appears in the records, baptized by them as an "epileptic recruiting rhythm" (ERR, at about 10 *Hz*). Later, as the seizure ends, they detected a progressive increase of lower frequencies associated with the convulsive phase. The TCES proceeds as follows: about 10 *s* after seizure onset, lower frequencies (0.5-3.5 *Hz*) are observed that gradually diminish their activity. The convulsive activity is associated to generalized polyspike bursts from muscle-jerks. Very slow irregular activity dominates then the EEG, accompanied with a gradual frequency increase of up to (3.5 - 12.5 *Hz*), indicative of the end of the seizure.

In Fig. 1.a we depict a typical EEG signal (sample frequency $\omega_s = 102.4$ *Hz*, for signal acquisition details see Ref. [2]) corre-