3) Both, trajectories and their two-time correlations obey an 'aging' scaling property typical of glassy dynamics when $\sigma \to 0$.
4) A progression from normal diffusiveness to subdiffusive behavior and finally to a stop in the growth of the mean square displacement as demonstrated by the use of a repeated-cell map. (see Fig. 3) The existence of this analogy is perhaps not accidental since the limit of vanishing noise amplitude $\sigma \to 0$ involves loss of ergodicity.

Localization & quasiperiodic onset of chaos

One interesting problem in condensed matter physics that exhibits connections with the quasiperiodic route to chaos is the localization transition for transport in incommensurate systems, where the discrete Schrödinger equation with a quasiperiodic potential translates into a nonlinear map known as the Harper map [14]. In this equivalence the divergence of the localization length corresponds to the vanishing of the ordinary Lyapunov coefficient. It is interesting to note that the basic features of q-statistics in the dynamics at critical attractors mentioned here turn up in the context of localization phenomena.

Acknowledgements

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[1] There are some parallels between the evolution towards an attractor in dissipative maps and the irreversible approach to equilibrium in thermal systems. It should be recalled that for the chaotic dynamics of a conservative system (such as the hard sphere gas) there is no dissipation nor strange attractor. It is of course this dynamics that is relevant to Boltzmann's assumption of molecular chaos.

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Nonextensive statistical mechanics and complex scale-free networks

Stefan Thurner
Complex Systems Research Group HNO, Medical University of Vienna, Währinger Gürtel 18-20, A-1090 Vienna, Austria

One explanation for the impressive recent boom in network theory might be that it provides a promising tool for an understanding of complex systems. Network theory is mainly focusing on discrete large-scale topological structures rather than on microscopic details of interactions of its elements. This viewpoint allows to naturally treat collective phenomena which are often an integral part of complex systems, such as biological or socio-economical phenomena. Much of the attraction of network theory arises from the discovery that many networks, natural or man-made, seem to exhibit some sort of universality, meaning that most of them belong to one of three classes: *random, scale-free* and *small-world* networks. Maybe most important however for the physics community is, that due to its conceptually intuitive nature, network theory seems to be within reach of a full and coherent understanding from first principles.

Networks are discrete objects made up of a set of nodes which are joint by a set of links. If a given set of N nodes is linked by a fixed number of links in a completely random manner, the result is a so-called *random network,* whose characteristics can be rather easily understood. One of the simplest measures describing a network in statistical terms is its degree distribution, $p(k)$, (see box 1). In the case of random networks the degree distribution is a Poissonian, i.e., the probability (density) that a randomly chosen node has degree k is given by $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, where $\lambda = \bar{k}$ is the average degree of all nodes in the network. However, as soon as more complicated rules for wiring or growing of a network are considered,

Some network measures

The degree k_i of a particular node i of the network is the number of links associated with it. If links are directed they either emerge or end at a node, yielding the diction of out- or in-degree, respectively. The degree distribution $p(k)$ is the probability for finding a node with degree k in the network. In (unweighted) networks the degree distribution is discrete and often reads, $p(k) = p(1)e_q^{k/\kappa} q$ with $\kappa > 0$ being some characteristic number of links. Apart from the degree distribution, important measures to characterize network topology are the clustering coefficient c_i, and the neighbor connectivity k_{nn}. The clustering coefficient measures the probability that two neighboring nodes of a node i are also neighbors of each other, and is thus a measure of cliquishness within networks. The neighbor connectivity is the average degree of all the nearest neighbors of node i. When plotted as a function of k, a non-trivial distribution of the average of c allows statements about hierarchic structures within the network, while k_{nn} serves as a measure of assortativity.

the seemingly simple concept of a network can become rather involved. In particular, in many cases the degree distribution becomes a power-law, bare of any characteristic scale, which raises associations to critical phenomena and scaling phenomena in complex systems. This is the reason why these types of networks are often called *complex networks.* A further intriguing aspect of dynamical complex networks is that they can potentially provide some sort of toy-model for 'nonergodic' systems, in the sense that not all possible states (configurations) are equally probable or homogeneously populated, and thus can violate a key assumption for systems described by classical statistical mechanics.

Over the past two decades the concept of *nonextensive statistical mechanics* has been extremely successful in addressing critical phenomena, complex systems and nonergodic systems [1, 2]. Nonextensive statistical mechanics is a generalization of Boltzmann-Gibbs statistical mechanics, where entropy is defined as

$$S_q \equiv \frac{1 - \int_0^\infty dk\, [p(k)]^q}{q-1} \qquad (1)$$

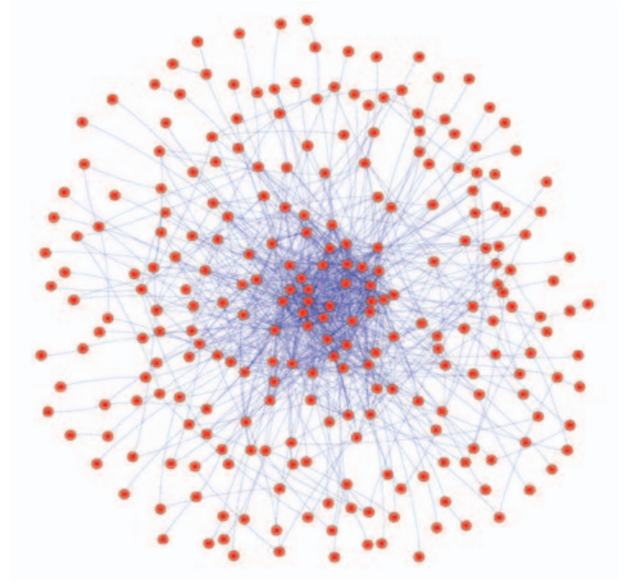
$$\text{with the } q=1 \text{ limit} \quad S_1 = S_{BG} \equiv -\int_0^\infty dk\, p(k) \ln p(k)$$

where *BG* stands for *Boltzmann-Gibbs.* If – in the philosophy of the maximum entropy principle – one extremizes S_q under certain constraints, the corresponding distribution is the q-exponential (see box in editorial paper by C. Tsallis and J.P. Boon). Another sign for the importance and ubiquity of q-exponentials in nature might be due to the fact that the most general Boltzman factor for canonical ensembles *(extensive)* is the q-exponential, as was recently proved in [3]. Given the above characteristics of networks and the fact that a vast number of real-world and model networks show asymptotic power-law degree distributions, it seems almost obvious to look for a connection between networks and nonextensive statistical physics.

Since the very beginning of the recent modeling efforts of complex networks it has been noticed that degree distributions asymptotically follow power-laws [4], or even exactly q-exponentials [5]. The model in [4] describes growing networks with a so-called preferential attachment rule, meaning that any new node being added to the system links itself to an already existing node i in the network with a probability that is proportional to the degree of node i, i.e. $p_{link} \propto k_i$. In [5] this model was extended to also allow for preferential rewiring. The analytical solution to the model has a q-exponential as a result, with the nonextensivity parameter q being fixed uniquely by the model parameters.

However, it has been found that networks exhibiting degree distributions compatible with q-exponentials are not at all limited to growing and preferentially organizing networks. Degree distributions of real-world networks as well as of models of various kinds seem to exhibit a universality in this respect. In the remainder we will review a small portion of the variety of networks which potentially have a natural link to non-extensive statistics. Even though there exists no complete theory yet, there is substantial evidence for a deep connection of complex networks with the $q \neq 1$ instance in nonextensive statistical mechanics.

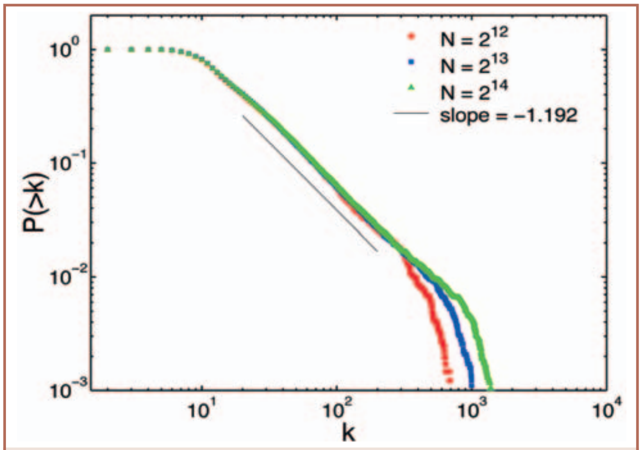
Recently in [6] preferential attachment networks have been embedded in Euclidean space, where the attachment probability for a newly added node is not only proportional to the degrees of existing nodes, but also depends on the Euclidean distance between nodes. The model is realized by setting the linking probability of a new node to an existing node i to be $p_{link} \propto k_i / r_i^\alpha (\alpha \geq 0)$, with r_i being the distance between the new node and node i; $\alpha = 0$ corresponds to the model in [4] which has no metrics. In a careful analysis the degree distributions of the resulting networks have been shown to be q-exponentials with a clear α-dependence of the nonextensivity parameter q. In the large α limit, q approaches one, i.e., random networks are recovered in the Boltzmann-Gibbs limit.



▲ **Fig. 1:** Snapshot of a non-growing dynamic network with q-exponential degree distribution for $N = 256$ nodes and a linking rate of $\bar{r} = 1$, for details see [8, 9]. The shown network is small to make connection patterns visible.

In an effort to understand the evolution of socio-economic networks, in [7] a model was proposed that builds upon [5] but introduces a rewiring scheme which depends on the *internal* network distance between two nodes, i.e., the number of steps needed to connect the two nodes. The emerging degree distributions have been subjected to a statistical analysis where the hypothesis of q-exponentials could not be rejected.

A model for nongrowing networks which was recently put forward in [8] also unambiguously exhibit q-exponential degree distributions. This model was motivated by interpreting networks as a certain type of 'gas' where upon an (inelastic) collision of two nodes, links get transfered in analogy to the energy-momentum transfer in real gases. In this model a fixed number of nodes in an (undirected) network can 'merge', i.e., two nodes fuse into one single node, which keeps the union of links of the two original nodes; the link connecting the two nodes before the merger is removed. At the same time a new node is introduced to the system and is linked randomly to any of the existing nodes in the network [9]. Due to the nature of this model the number of links is not strictly conserved – which can be thought of as jumps between discrete states in some 'phase space'. The model has been further generalized to exhibit a distance dependence as in [6], however r_i not being Euclidean but internal distance. Again, the resulting degree distributions have q-exponential form. In Fig. 1 we show a snapshot of this type of network *pars pro toto* for the many models exhibiting q-exponential degree distributions. The corresponding (cumulative) degree distribution is shown in Fig. 2 in log-log scale, clearly exhibiting a power-law. Figure 3 shows q-logarithms of the degree distribution for several values of q. It is clear from the correlation coefficient of the q-logarithm with straight lines (inset) that there exists an optimal value of q, which makes the q-logarithm a linear function in k, showing that the degree distribution is a q-exponential.



▲ **Fig. 2:** Log-log representation of (cumulative) degree distributions for the same type of network with various system sizes N and $\bar{r} = 8$. The distribution functions are from individual network configurations without averaging over identical realizations. On the right side the typical exponential finite size effect is seen.

A quite different approach was taken in [10] where an ensemble interpretation of random networks has been adopted, motivated by superstatistics [11]. Here it was assumed that the average connectivity λ in random networks is fluctuating according to a distribution $\Pi(\lambda)$, which is sometimes associated with a 'hidden-variable' distribution. In this sense a network with any degree distribution can be seen as a 'superposition' of random networks with the degree distribution given by $p(k) = \int_0^\infty d\lambda \Pi(\lambda) \frac{\lambda^k e^{-\lambda}}{k!}$. In [10] it was shown as an exact example, that a power-law functional form of $\Pi(\lambda)$ leads to degree distributions of Zipf-Mandelbrot form, $p(k) \propto \frac{1}{(k_0+k)^\gamma}$, which is equivalent to a q-exponential with an argument of $k/\mathcal{K}$ and given the substitutions, $\mathcal{K} = (1 - q)k_0$ and $q = 1 + 1/\gamma$. Very recently a possible connection between *small-world* networks and the maximum Tsallis-entropy principle, as well as to the hidden variable method [10], has been noticed in [12].

In yet another view of networks from a physicist's perspective, networks have recently been treated as statistical systems on a Hamiltonian basis. It has been shown that these systems show a phase transition like behavior [13], along which network structure changes. In the low temperature phase one finds networks of 'star' type, meaning that a few nodes are extremely well connected resulting even in a discontinuous $p(k)$; in the high temperature phase one finds random networks. Surprisingly, for a special type of Hamiltonian, networks with q-exponential degree distributions emerge right in the vicinity of the transition point [14].

While a full theory of how complex networks are connected to $q \neq 1$ statistical mechanics is still missing, it is almost clear that such a relation should exist. It would not be surprising if an understanding of this relation would arise from the very nature of networks, being *discrete objects*. More specifically, it has been conjectured for nonextensive systems that the microscopic dynamics does not fill or cover the space of states (e.g. Γ-space ($6N$ dimensional phase space) for Hamiltonian systems) in a homogeneous and equi-probable manner [2]. This possibly makes phase space for nonextensive systems look like a network itself, in the sense that in a network not all possible positions in space can be taken, but that microscopic dynamics is restricted onto nodes and links. In this view the basis of nonextensive systems could be connected to a network like structure of their 'phase space'. It would be fantastic if further understanding of network theory could propel a deep understanding of nonextensive statistical physics, and vice versa, making them co-evolving theories.

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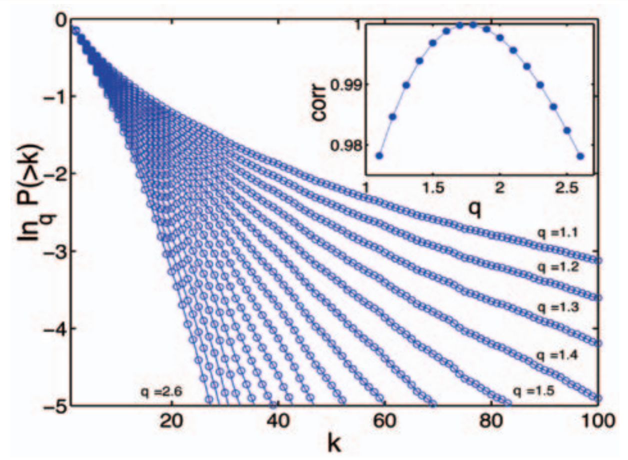

▲ **Fig.3:** q-logarithm of the (cumulative) distribution function from the previous figure as a function of degree k. Clearly, there exists an optimum q which allows for an optimal linear fit. *Inset:* Linear correlation coefficient of $\ln_q P(\geq k)$ and straight lines for various values of q. The optimum value of q is obtained when $\ln_q P(\geq k)$ is optimally linear, i.e., where the correlation coefficient has a maximum. A linear $\ln_q$ means that the distribution function is a q-exponential; the slope of the linear function determines $\mathcal{K}$. In this example we get for the optimum $q = 1.84$, which corresponds to the slope $\gamma = 1.19$ in the previous plot.