

momentum distribution can be different in the high energy tail. In fact, this quantum uncertainty effect (not Heisenberg uncertainty) between energy \mathcal{E} and momentum p , caused by the many-body collisions and described by the Kadanoff-Baym equation, implies an energy-momentum distribution of the form

$$f_Q(\mathcal{E}, p) = \frac{1}{\pi} n(\mathcal{E}) \delta_\gamma(\mathcal{E}, p) \quad (6)$$

with

$$\delta_\gamma(\mathcal{E}, p) = \frac{Im\Sigma^R(\mathcal{E}, p)}{(\mathcal{E} - \mathcal{E}_p - Re\Sigma^R(\mathcal{E}, p))^2 + (Im\Sigma^R(\mathcal{E}, p))^2} \quad (7)$$

where $\Sigma^R(\mathcal{E}, p)$ is the mass operator of the one-particle Green function. After integrating in $d\mathcal{E}$ the product of $f_Q(\mathcal{E}, p)$ and the Maxwellian energy distribution, we obtain a momentum distribution with an enhanced high-momentum tail. Although this approach produces a deviation from the MB distribution, the state represented by $f_Q(p)$ is an equilibrium state [11, 12]. The Maxwellian distribution is recovered in the limit when $\delta_\gamma(\mathcal{E}, p)$ becomes a δ function with a sharp correspondence between momentum and energy.

Distributions different from the Maxwellian one can also be obtained axiomatically from non-standard, but mathematically consistent, versions of statistical mechanics that use entropies different from the Boltzmann-Gibbs one [4, 13].

We have argued that it is not sufficient to know that the Maxwellian distribution is a very good approximation to the particle distribution. We must be sure that there are no corrections to a very high accuracy, when studying reactions that are highly sensitive to the tail of the distribution, such as fusion reactions between charged ions. Several mechanisms have been outlined (others need to be studied) that can produce small, but important deviations in the tail of the distribution.

References

- [1] R. d'Escourt Atkinson and F. G. Houtermans, *Z. Physik* **54** (1929) 656.
- [2] V. Castellani et al, *Phys. Rept.* **281** (1997) 309 [arXiv:astro-ph/9606180].
- [3] D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis* (New York: McGraw-Hill Book Company, 1968).
- [4] C. Tsallis, *J. Statist. Phys.* **52** (1988) 479.
- [5] M. Coraddu et al, *Physica A* **305** (2002) 282 [arXiv:physics/0112018].
- [6] S. Degl'Innocenti et al, *Phys. Lett. B* **441** (1998) 291 [arXiv:astro-ph/9807078].
- [7] M. Coraddu et al, *Braz. J. Phys.* **29** (1999) 153 [arXiv:nucl-th/9811081].
- [8] G. Kaniadakis and P. Quarati, *Physica A* **192** (1993) 677; *ibid.* **237** (1997) 229.
- [9] F. Ferro and P. Quarati, *Phys. Rev. E* **71** (2005) 026408 [arXiv:cond-mat/0407665].
- [10] V. M. Galitskii and V. V. Yakimets, *JEPT* **24** (1967) 637.
- [11] A. N. Starostin, V. I. Savchenko, and N. J. Fisch, *Phys. Lett. A* **274** (2000) 64.
- [12] M. Coraddu et al, *Physica A* **340** (2004) 490 [arXiv:nucl-th/0401043].
- [13] G. Kaniadakis et al, *Phys. Rev. E* **71** (2005) 046128 [arXiv:cond-mat/0409683].

Critical attractors and q -statistics

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The singular dynamics at critical attractors of even the simplest one dimensional nonlinear iterated maps is of current interest to statistical physicists because it provides insights into the limits of validity of the Boltzmann-Gibbs (BG) statistical mechanics. This dynamics also helps inspect the form of the possible generalizations of the canonical formalism when its crucial supports, phase space mixing and ergodicity, break down.

The fame of the critical attractors present at the onset of chaos in the logistic and circle maps stems from their universal properties, comparable to those of critical phenomena in systems with many degrees of freedom. At these attractors the indicators of chaos withdraw, such as the fast rate of separation of initially close by trajectories. As it is generally understood, the standard exponential divergence of trajectories in chaotic attractors suggests a mechanism to justify the property of irreversibility in the BG statistical mechanics [1]. In contrast, the onset of chaos imprints memory preserving properties to its trajectories.

The dynamical nature of trajectories is appraised on a regular basis through the sensitivity to initial conditions ξ_t , defined as

$$\xi_t(x_0) \equiv \lim_{\Delta x_0 \rightarrow 0} (\Delta x_t / \Delta x_0), \quad t \text{ large}, \quad (1)$$

where Δx_0 is the initial separation of two trajectories and Δx_t that at time t . For a one-dimensional map it has the form $\xi_t(x_0) = \exp(\lambda_1 t)$, with $\lambda_1 > 0$ for chaotic attractors and $\lambda_1 < 0$ for periodic ones. The number λ_1 is called the Lyapunov coefficient. At critical attractors $\lambda_1 = 0$ and ξ_t does not settle onto a single-valued function but exhibits instead fluctuations that grow indefinitely. For initial positions on the attractor ξ_t develops a universal self-similar temporal structure and its envelope grows with t as a power law.

It has been recently corroborated [2]-[5] that the dynamics at the critical attractors associated with the three familiar routes to chaos, intermittency, period doubling and quasiperiodicity [6], obey the features of the q -statistics, the generalization of BG statistics based on the q -entropy S_q [7]. The focal point of the q -statistical description for the dynamics at such attractors is a ξ_t associated with one or several expressions of the form

$$\xi_t(x_0) = \exp_q[\lambda_q(x_0) t], \quad (2)$$

where q is the entropic index and λ_q is the q -generalized Lyapunov coefficient. Also the identity $K_1 = \lambda_1$ (where the rate of entropy production K_1 is given by $K_1 t = S_{BG}(t) - S_{BG}(0)$ with S_{BG} the Boltzmann-Gibbs entropy) generalizes to

$$K_q = \lambda_q, \quad (3)$$

where the rate of q -entropy production K_q is defined via $K_q t = S_q(t) - S_q(0)$ [3], [4], [7].

Tsallis q index & Mori's q -phase transitions

The central issue of research in q -statistics is perhaps to confirm the occurrence of special values for the entropic index q for any given system and to establish their origin. In the case of critical

attractors the allowed values for q are obtained from the universality class parameters to which the attractor belongs. For the simpler cases, the pitchfork and the tangent bifurcations, there is a single well-defined value for the index q [2]. The pitchfork bifurcations form the sequence of period doublings that culminate in the chaos threshold whereas at the tangent bifurcation chaos develops via intermittency. For critical multifractal attractors, as in the period doubling accumulation point and in the quasiperiodic onset of chaos, the situation is more complicated and there appear to be a multiplicity of indexes q but with precise values given by the attractor scaling functions. They come out in pairs and are related to the occurrence of dynamical ‘ q -phase’ transitions [4], and these are identified as the source of the special values for the entropic index q . The q -phase transitions connect qualitatively different regions of the multifractal attractor.

The main quantities in the thermodynamic formalism of q -phase transitions (developed by Mori and colleagues in the late 80’s [8]) are the spectral functions $\phi(q)$ and $\psi(\lambda)$, related to each other via Legendre transformation, the function of generalized Lyapunov coefficients $\lambda(q)$, given by $\lambda(q) \equiv d\phi(q)/dq$, and the variance $\nu(q) \equiv d\lambda(q)/dq$. The functions $\phi(q)$ and $\psi(\lambda)$ are the dynamic counterparts of the multifractal dimensions $D(q)$ and the spectrum $f(\alpha)$ that characterize the geometric structure of the attractor [6]. As with ordinary thermal 1st order phase transitions, a q -phase transition is indicated by a section of linear slope $m_c = 1 - q$ in the ‘free energy’ $\psi(\lambda)$, a discontinuity at $q = q$ in the ‘order parameter’ $\lambda(q)$, and a divergence at q in the ‘susceptibility’ $\nu(q)$. Actually an infinite family of such transitions takes place but of rapidly decreasing strengths [4], [5].

Tangent & pitchfork bifurcations

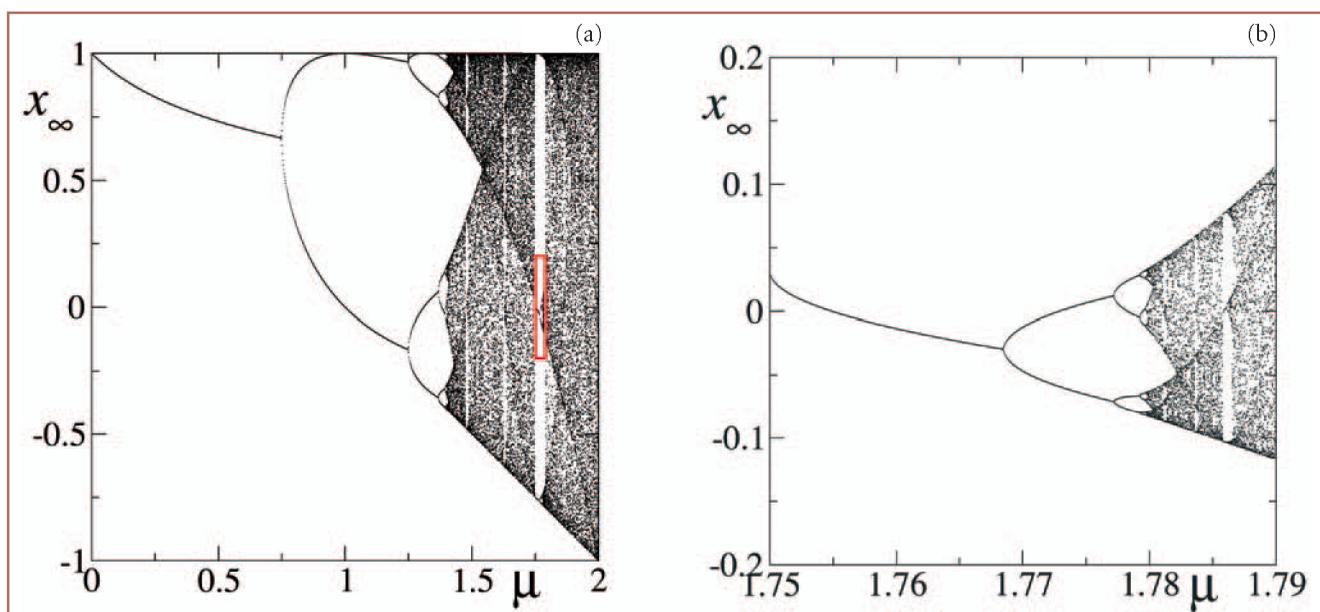
The tangent bifurcations of unimodal (one hump) maps of general nonlinearity $z > 1$ display weak insensitivity to initial conditions, i.e. power-law convergence of orbits when at the left-hand side ($x < x_c$) of the point of tangency x_c . However at the right-hand side ($x > x_c$) of the bifurcation there is a ‘super-strong’ sensitivity to initial conditions, i.e. a sensitivity that grows faster than exponential [2]. The two different behaviors can be couched

as a q -phase transition with indexes q and $2 - q$ for the two sides of the tangency point. The pitchfork bifurcations display weak insensitivity to initial conditions.

For the tangent bifurcations one has always $q = 3/2$, while for the pitchfork bifurcation one has $q = 5/3$. Notably, these results for the index q are valid for all $z > 1$ and therefore define the existence of only two universality classes for unimodal maps [2]. The treatment of the tangent bifurcation differs from other studies of intermittency transitions in that there is no feedback mechanism of iterates from an adjacent chaotic region. Therefore, impeded or incomplete mixing in phase space (a small interval neighbourhood around $x = x_c$) arises from the special ‘tangency’ shape of the map at the transitions that produces monotonic trajectories. This has the effect of confining or expelling trajectories causing anomalous phase-space sampling, in contrast to the thorough coverage in generic states with $\lambda_1 > 0$.

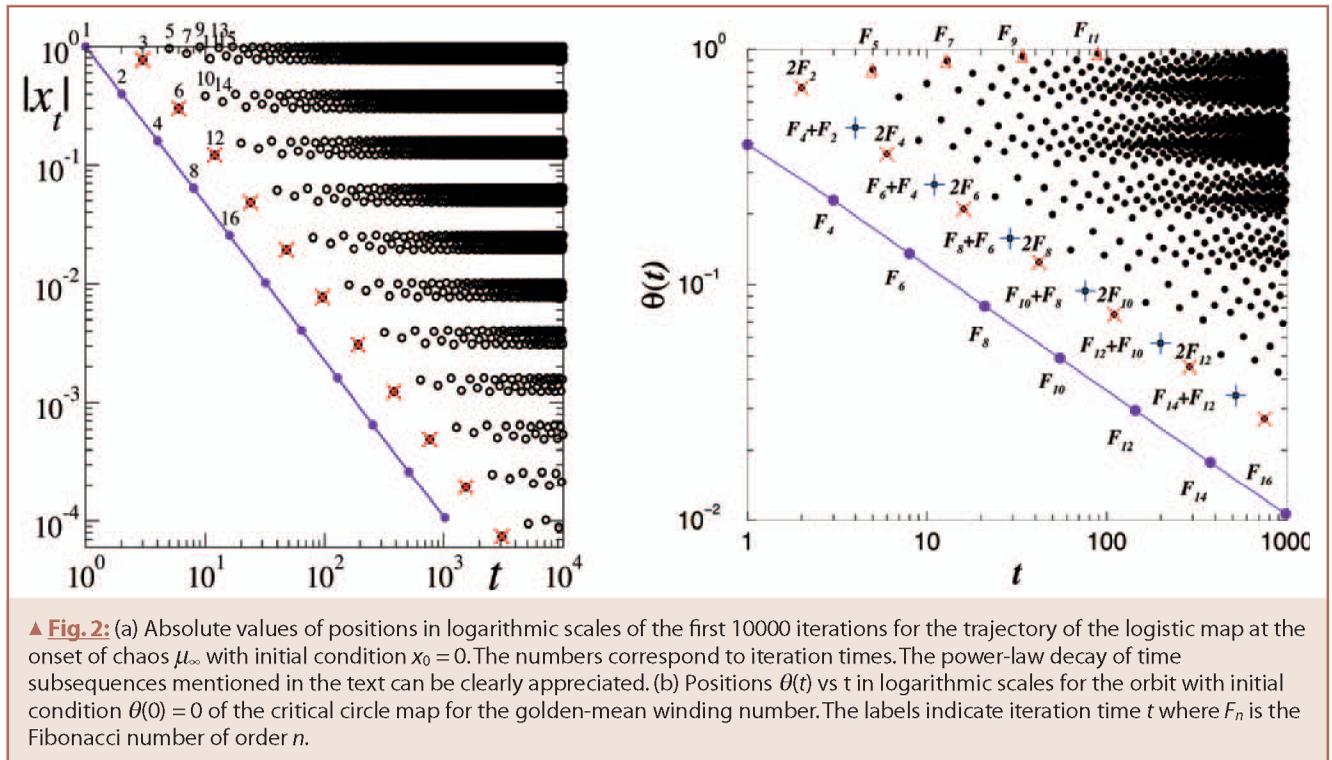
Period-doubling accumulation point

For a unimodal map of nonlinearity $z > 1$ (e.g. the logistic map has $z = 2$) with extremum at $x = 0$ and control parameter μ the onset of chaos is obtained at the accumulation point μ_∞ of the μ values for the pitchfork bifurcations $\mu_n, n = 1, 2, \dots$, [6]. This is often called the Feigenbaum attractor which reappears in multiples together with the precursor cascade of period-doubling bifurcations in the infinite number of windows of periodic trajectories that interpose the chaotic attractors beyond μ_∞ . The number of cascades within each window is equal to the period of the orbit that emerges at the tangent bifurcation at its opening. See Fig. 1. The dynamics at the Feigenbaum attractor has been analyzed recently [3], [4]. By taking as initial condition $x_0 = 0$ at μ_∞ it is found that the resulting orbit of period 2^∞ consists of trajectories made of intertwined power laws that asymptotically reproduce the entire period-doubling cascade that occurs for $\mu < \mu_\infty$ (see Fig. 2b) It was established that ξ_r has precisely the form of a set of interlaced q -exponentials, of which the q -indexes and the sets of associated q -Lyapunov coefficients λ_q were determined. As mentioned, the appearance of a specific value for the q index (and actually also that for its conjugate value $Q = 2 - q$) turns out to be



▲ Fig. 1: (a) The classic logistic map attractor as a function of control parameter μ . (b) Enlargement of the box in (a).

features



due to the occurrence of q -phase transitions between ‘local attractor structures’ at μ_∞ . The values of the q -indexes are obtained from the discontinuities of the universal trajectory scaling function σ . This function characterizes the multifractal by measuring how adjacent positions of orbits of period 2^n approach each other as $n \rightarrow \infty$ [9]. The main discontinuity in σ is related to the most crowded and most sparse regions of the attractor and in this case $q = 1 - \ln 2 / (z - 1) \ln \alpha_F$, where α_F is the universal scaling constant associated with these two regions. Furthermore, it has also been shown [3], [4] that the dynamical and entropic properties at μ_∞ are naturally linked through the q -exponential and q -logarithmic expressions, respectively, for the sensitivity ξ_t and for the entropy S_q in the rate of entropy production K_q .

Quasiperiodicity & golden mean route to chaos

A recent study [5] of the dynamics at the quasiperiodic onset of chaos in maps with zero slope inflection points of cubic nonlinearity (e.g. the critical circle map) shows strong parallelisms with the dynamics at the Feigenbaum attractor described above. Progress on detailed knowledge about the structure of the orbits within the golden-mean quasiperiodic attractor, see Fig. 2b, helped determine the sensitivity to initial conditions for sets of starting positions within this attractor [5]. It was found that ξ_t is made up of a hierarchy of families of infinitely many interconnected q -exponentials. Here again, each pair of regions in the multifractal attractor, that contain the starting and finishing positions of a set of trajectories, leads to a family of q -exponentials with a fixed value of the index q and an associated spectrum of q -Lyapunov coefficients λ_q .

As in the period doubling route to chaos, the quasiperiodic dynamics consists of an infinite family of q -phase transitions, each associated to trajectories that have common starting and finishing positions located at specific regions of the multifractal. The specific values of the variable q (in the thermodynamic formalism) at which the q -phase transitions take place are the same values for the entropic index q in ξ_t . The transitions come in pairs at q and

$2 - q$ as they are tied down to the expressions for λ_q in ξ_t . Again, the dominant dynamical transition is associated to the most crowded and sparse regions of the multifractal, and the value of its q -index is [5] $q = 1 - \ln w_{gm} / 2 \ln \alpha_{gm}$ where $w_{gm} = (\sqrt{5} - 1) / 2$ is the golden mean and α_{gm} is the universal constant that plays the same scaling role as α_F .

Structure in dynamics

The dynamical organization within critical multifractal attractors is difficult to resolve from the consideration of a straightforward time evolution, i.e. the record of positions at every time t for a trajectory started at an arbitrary position x_0 within the attractor. In this case what is observed are strongly fluctuating quantities that persist in time with a scrambled pattern structure that exhibits memory retention. Unsystematic averages over x_0 would rub out the details of the multiscale dynamical properties we uncovered. On the other hand, if specific initial positions with known location within the multifractal are chosen, and subsequent positions are observed only at pre-selected times, when the trajectories visit another selected region, a distinct q -exponential expression for ξ_t is obtained.

Manifestations of q -statistics in condensed matter problems

There are connections between the properties of critical attractors referred to here and those of systems with many degrees of freedom at extremal or transitional states. Three specific examples have been recently developed (see Table). In one case the dynamics at the chaos threshold via intermittency has been shown to be related to that of intermittent clusters at thermal critical states [10]. In the second case the dynamics at the noise-perturbed period-doubling onset of chaos has been found to show parallelisms with the glassy dynamics observed in supercooled molecular liquids [11]. In the third case the known connection between the quasiperiodic route to chaos and the localization transition for transport in incommensurate systems is analyzed from the perspective of the q -statistics [12].

Critical clusters & intermittency

The dynamics of fluctuations of an equilibrium critical state in standard models of a magnetic or fluid system is seen to be related to the dynamics at a critical attractor for intermittency. The connection can be examined when instead of the entire critical system a single unstable cluster of excess magnetization or density of size R is considered. The analysis, initially developed by Contoyiannis and colleagues [13], has been reconsidered recently in connection with q -statistics [10].

An important element in the analysis is the determination of the order parameter $\phi(r)$ of a large cluster by withholding only its most probable configurations from a coarse-grained partition function Z . The conditions under which these configurations dominate are evaluated as these determine an instability of the cluster. The instability can be expressed as an inequality between two lengths in space. This is $r_0 \gg R$, where r_0 is the location of a divergence in the expression for $\phi(r)$. In a coarse-grained time scale the cluster is expected to evolve by increasing its average amplitude $\bar{\phi}$ and/or size R because the subsystem studied represents an environment with unevenness in the states of the microscopic degrees of freedom (e.g. more spins up than down). Increments in $\bar{\phi}$ for fixed R takes the position r_0 for the singularity closer to R , the dominance of this configuration in Z decreases accordingly and rapidly so. When the divergence is reached at $r_0 = R$ the profile $\phi(r)$ no longer describes the spatial region where the subsystem is located. But a subsequent fluctuation would again be represented by a cluster $\phi(r)$ of the same type. From this renewal process we obtain a picture of intermittency [10].

Amongst the static and dynamical properties for such single critical cluster of order parameter $\phi(r)$ we mention [10]: 1) The faster than exponential growth with cluster size R of the space-integrated ϕ suggests nonextensivity of the BG entropy but extensivity of a q -entropy expression. 2) The finding that the time evolution of ϕ is described by the dynamics of the critical attractor for intermittency which implies an atypical sensitivity to initial conditions compatible with q -statistics. 3) Both, the approach to criticality and the infinite-size cluster limit at criticality manifest through a crossover from canonical to q -statistics.

Glassy dynamics & noise-perturbed Feigenbaum attractor

The erratic motion of a Brownian particle is usually described by the Langevin theory. As it is well known, this method finds a way round the detailed consideration of many degrees of freedom by representing the effect of collisions with molecules in the fluid in

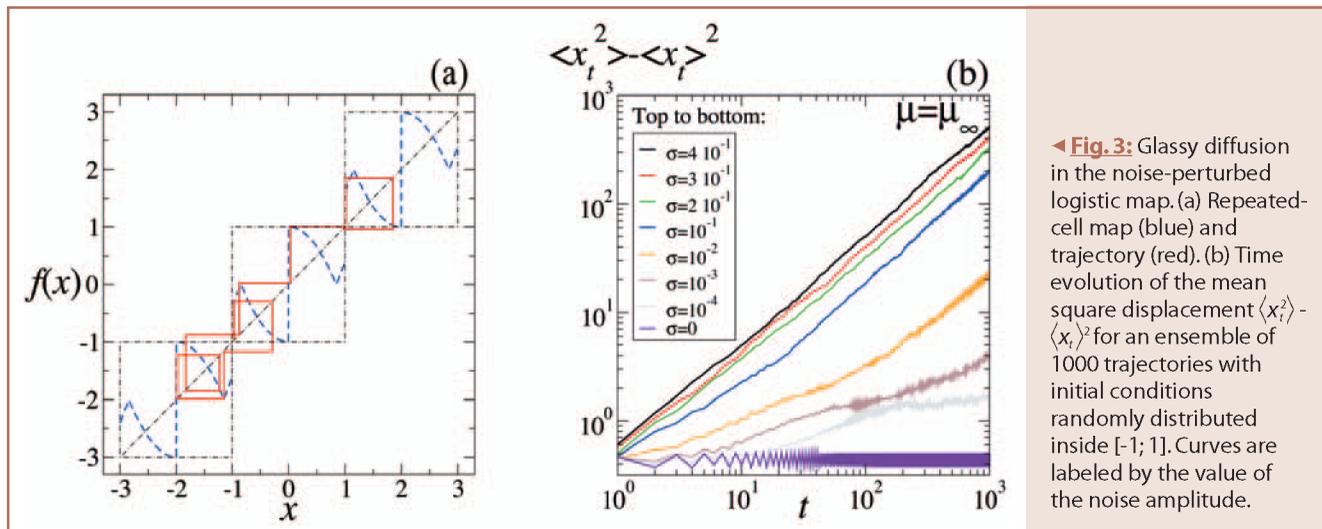
Route to chaos	Intermittency	Period doubling	Quasiperiodicity
Common properties	Vanishing ordinary Lyapunov coefficient, dynamical phase transitions (Mori's q -phases) power-law dynamics, q -sensitivity, q -Pesin identity		
Distinctive properties	(Also) faster than exponential dynamics	Foam-like phase space	Dense phase space
Applications in condensed matter physics	Critical clusters	Glass formation	Localization
Applications in other disciplines	Information & other flows in networks, ...	Protein folding, vegetation patterns, ...	Mode locking, cardiac cells, Internet TCP, ...

▲ **Table:** Summary. The three routes to chaos, properties and applications.

which the particle moves by a noise source. The approach to thermal equilibrium is produced by random forces, and these are sufficient to determine dynamical correlations, diffusion, and a basic form for the fluctuation-dissipation theorem. In the same spirit, attractors of nonlinear low-dimensional maps under the effect of external noise can be used to model states in systems with many degrees of freedom. In a one-dimensional map with only one control parameter μ the consideration of external noise of amplitude σ could be thought to represent the effect of many other systems coupled to it, like in the so-called coupled map lattices [9]. The general map formula can be seen to represent a discrete form for a Langevin equation with nonlinear 'friction force' term [11].

The dynamics of noise-perturbed logistic maps at the chaos threshold has been shown to exhibit parallels with the most prominent features of glassy dynamics in, for example, supercooled liquids. In this analogy the noise amplitude σ plays a role similar to the temperature difference from a glass transition temperature. Specifically our results are [11]:

- 1) Two-step relaxation occurring in trajectories and in their two-time correlations when $\sigma \rightarrow 0$.
- 2) A map equivalent to a relationship between the relaxation time and the configurational entropy.



◀ **Fig. 3:** Glassy diffusion in the noise-perturbed logistic map. (a) Repeated-cell map (blue) and trajectory (red). (b) Time evolution of the mean square displacement $\langle x_t^2 \rangle - \langle x_r \rangle^2$ for an ensemble of 1000 trajectories with initial conditions randomly distributed inside $[-1; 1]$. Curves are labeled by the value of the noise amplitude.

3) Both, trajectories and their two-time correlations obey an 'aging' scaling property typical of glassy dynamics when $\sigma \rightarrow 0$.
 4) A progression from normal diffusiveness to subdiffusive behavior and finally to a stop in the growth of the mean square displacement as demonstrated by the use of a repeated-cell map. (see Fig. 3) The existence of this analogy is perhaps not accidental since the limit of vanishing noise amplitude $\sigma \rightarrow 0$ involves loss of ergodicity.

Localization & quasiperiodic onset of chaos

One interesting problem in condensed matter physics that exhibits connections with the quasiperiodic route to chaos is the localization transition for transport in incommensurate systems, where the discrete Schrödinger equation with a quasiperiodic potential translates into a nonlinear map known as the Harper map [14]. In this equivalence the divergence of the localization length corresponds to the vanishing of the ordinary Lyapunov coefficient. It is interesting to note that the basic features of q -statistics in the dynamics at critical attractors mentioned here turn up in the context of localization phenomena.

Acknowledgements

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References

- [1] There are some parallels between the evolution towards an attractor in dissipative maps and the irreversible approach to equilibrium in thermal systems. It should be recalled that for the chaotic dynamics of a conservative system (such as the hard sphere gas) there is no dissipation nor strange attractor. It is of course this dynamics that is relevant to Boltzmann's assumption of molecular chaos.
- [2] F. Baldovin and A. Robledo, *Europhys. Lett.* **60**, 518 (2002); A. Robledo, *Physica D* **193**, 153 (2004).
- [3] F. Baldovin and A. Robledo, *Phys. Rev. E* **66**, 045104 (2002); *E* **69**, 045202 (2004).
- [4] E. Mayoral and A. Robledo, *Phys. Rev. E* **72**, 026209 (2005).
- [5] H. Hernández-Saldaña and A. Robledo, *Phys. Rev. E* (submitted) and cond-mat/0507624.
- [6] See, for example, R.C. Hilborn, *Chaos and nonlinear dynamics*, 2nd Revised Edition (Oxford University Press, New York, 2000).
- [7] C. Tsallis, A.R. Plastino and W.-M. Zheng, *Chaos, Solitons and Fractals* **8**, 885 (1997).
- [8] H. Mori, H. Hata, T. Horita and T. Kobayashi, *Prog. Theor. Phys. Suppl.* **99**, 1 (1989).
- [9] See, for example, H.G. Schuster and W. Just, *Deterministic Chaos. An Introduction*, 4th Revised and enlarged Edition (John Wiley & Sons Publishers, 2005).
- [10] A. Robledo, *Physica A* **344**, 631 (2004); *Molec. Phys.* **103**, 3025 (2005).
- [11] A. Robledo, *Phys. Lett. A* **328**, 467 (2004); F. Baldovin and A. Robledo, *Phys. Rev. E* (in press) and cond-mat/0504033.
- [12] H. Hernández-Saldaña and A. Robledo, in preparation.
- [13] Y.F. Contoyiannis and F.K. Diakonou, *Phys. Lett. A* **268**, 286 (2000); Y.F. Contoyiannis, F.K. Diakonou, and A. Malakis, *Phys. Rev. Lett.* **89**, 035701 (2002). 7
- [14] J.A. Ketoja and I.I. Satija, *Physica D* **109**, 70 (1997).

Nonextensive statistical mechanics and complex scale-free networks

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One explanation for the impressive recent boom in network theory might be that it provides a promising tool for an understanding of complex systems. Network theory is mainly focusing on discrete large-scale topological structures rather than on microscopic details of interactions of its elements. This viewpoint allows to naturally treat collective phenomena which are often an integral part of complex systems, such as biological or socio-economical phenomena. Much of the attraction of network theory arises from the discovery that many networks, natural or man-made, seem to exhibit some sort of universality, meaning that most of them belong to one of three classes: *random*, *scale-free* and *small-world* networks. Maybe most important however for the physics community is, that due to its conceptually intuitive nature, network theory seems to be within reach of a full and coherent understanding from first principles.

Networks are discrete objects made up of a set of nodes which are joint by a set of links. If a given set of N nodes is linked by a fixed number of links in a completely random manner, the result is a so-called *random network*, whose characteristics can be rather easily understood. One of the simplest measures describing a network in statistical terms is its degree distribution, $p(k)$, (see box 1). In the case of random networks the degree distribution is a Poissonian, i.e., the probability (density) that a randomly chosen node has degree k is given by $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, where $\lambda = \bar{k}$ is the average degree of all nodes in the network. However, as soon as more complicated rules for wiring or growing of a network are considered,

Some network measures

The degree k_i of a particular node i of the network is the number of links associated with it. If links are directed they either emerge or end at a node, yielding the diction of out- or in-degree, respectively. The degree distribution $p(k)$ is the probability for finding a node with degree k in the network. In (unweighted) networks the degree distribution is discrete and often reads, $p(k) = p(1) e^{-\kappa} \kappa^k$ with $\kappa > 0$ being some characteristic number of links. Apart from the degree distribution, important measures to characterize network topology are the clustering coefficient c_i , and the neighbor connectivity k_{nn} . The clustering coefficient measures the probability that two neighboring nodes of a node i are also neighbors of each other, and is thus a measure of cliquishness within networks. The neighbor connectivity is the average degree of all the nearest neighbors of node i . When plotted as a function of k , a non-trivial distribution of the average of c allows statements about hierarchic structures within the network, while k_{nn} serves as a measure of assortativity.