regime is fixed. As expected, the model is nonergodic and, accordingly, the time average and the ensemble average of a physical quantity are different from each other. We have ascertained that, for the event-event correlation function (defined by the natural-time average) in Eq. (1), the model well reproduces the aging and scaling properties (together with the form of the scaling function, $\tilde{C}$) discovered in Ref.[3] for real seismicity. On the other hand, an intriguing feature was found for the event-event correlation function defined by the ensemble average with respect to numerical runs. To distinguish such a correlation function from the one with the natural-time averages in Eq. (1), it is denoted here by $D(n + n_w, n_w)$. This quantity also turned out to exhibit the aging phenomenon with respect to the natural waiting time, that is, *the smaller the value of the natural waiting time is, the faster correlation decays.* So, the system has an internal clock. Fig. 1 shows that the aging curves become collapsed to a single curve by the rescaling of the natural time: $D(n + n_w, n_w) = \tilde{D}(n/(n_w)^{1.05})$, establishing the scaling property. The inset presents its semi-q-log plot. The straight line there implies that the scaling function, $\tilde{D}$, is given by the q-exponential function.

Now, according to Tsallis [6], there may be "q-triplet" $\{q_{stat}, q_{sen}, q_{rel}\}$ for a system described by nonextensive statistics, where q_{stat} is the entropic index appearing in the maximum Tsallis entropy distribution as well as the Tsallis entropy itself, q_{sen} is the index characterizing sensitivity of a nonlinear dynamical system to the initial condition and q_{rel} controls the rate of relaxation and decay of correlation. In the case of a simple system described by Boltzmann-Gibbs-type statistics, the q-triplet may be given by $\{q_{stat}, q_{sen}, q_{rel}\} = \{1, 1, 1\}$. In the case of a complex system, physical quantities are often expressed empirically in terms of the q-exponential function: e.g., probability distributions (q_{stat}), the distance between two trajectories of a dynamical system (q_{sen}) and relaxation or correlation (q_{rel}), with the values all different from unity. An example is provided by the recent work done by the people from NASA [7], who have discovered a non-Boltzmann-Gibbs case in a single physical setup. Analyzing the fluctuating magnetic field strength observed by Voyager 1 in the solar wind, they have found that $\{q_{stat}, q_{sen}, q_{rel}\} = \{1.75 \pm 0.06, -0.6 \pm 0.2, 3.8 \pm 0.3\}$.

Therefore, the q-exponential scaling function obtained for aftershocks in the coherent noise model, $\tilde{D}$, with $q_{rel} \simeq 2.98$, which is notably different from unity, could be seen as a fingerprint of further relevance of nonextensive statistics.

Science of complexity certainly enables one to reveal novel aspects of real seismicity. Nonextensive statistics is expected to offer a guiding principle for a deeper understanding of complex dynamical systems with catastrophes, in general and complexity of seismicity, in particular.

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S_q entropy and self-gravitating systems

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More than fifteen yeas ago, a generalized thermostatistical formalism (usually referred to as nonextensive statistical mechanics) based on a power-law entropic measure S_q was advanced by Constantino Tsallis [1]. Increasing attention has been paid to the Tsallis proposal in recent years, because it has been hailed by many researchers as a useful tool for the description of certain aspects of physical scenarios exhibiting atypical thermodynamical features due, for instance, to the presence of long range interactions. When the Tsallis formalism appeared in 1988, it was not at all clear what possible physical applications it might have. The first hint pointing towards a relationship between the Tsallis' ideas and the thermodynamics of systems with long range interactions came in 1993 [2], when it was realized that the Tsallis entropic functional is closely related to a family of distribution functions well known by astronomers studying the dynamics of stellar systems: the polytropic stellar distributions. In point of fact, the discovery of the connection between Tsallis entropy and the stellar polytropic distribution constituted the first application of the Tsallis' formalism to a concrete physical problem. Stellar systems, such as stellar clusters or galaxies, are important examples of astrophysical self-gravitating systems, where the gravitational interaction between the constituents of the system play a fundamental role in determining the system's properties. The exploration of the connections between the Tsallis thermostatistical formalism and the physics of self-gravitating systems has been the focus of a considerable research activity in recent years [3-7].

Nonextensive statistical mechanics is built up on the basis of the nonextensive, power-law entropy S_q [1]. The entropic index q (also called the Tsallis' entropic parameter) characterizes the statistics we are dealing with. In the limit $q \to 1$ the usual Boltzmann Gibbs (BG) expression is recovered: $S_1 = S_{BG}$ (for the definitions of the basic quantities associated with the q-entropy see the Box in the editorial introduction by Tsallis and Boon to the present Issue). Optimizing the entropic measure S_q under the constraints imposed by normalization and the mean value of the energy, one obtains the probability distribution associated (in the context of Tsallis' formalism) with the thermal equilibrium or metaequilibrium of the system under consideration. The main property of this q-generalized maximum entropy distribution is that it exhibits a power-law like dependence on the microstate energy ϵ_i, instead of the exponential dependence associated with the standard Boltzmann-Gibbs thermostatistics.

Galaxies can be regarded as self gravitating N-body systems that are trapped for a long time in a non-collisional regime (until collisional effects finally become important) where the stars move under the influence of the mean potential Φ generated by the whole set of stars, Φ being a function of the spatial position $\mathbf{x}$.

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The description of a state of any collisionless system is given by a distribution function $F(\mathbf{x}, \mathbf{v}, t)$ in a 6-dimensional phase space where $\mathbf{v}$ is the velocity vector and t the time. The fundamental equation of stellar dynamics is (as far as collisionless systems are concerned) the Vlasov equation

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla_x F - \nabla_x \Phi \cdot \nabla_v F = 0, \tag{1}$$

coupled to an equation relating the distribution function $F(\mathbf{x}, \mathbf{v}, t)$ with the Newtonian gravitational potential Φ,

$$\Phi(\mathbf{x}) = -Gm \int \frac{F(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \, d^3\mathbf{v}', \tag{2}$$

where G is Newton's gravitational constant and m denotes the mass of each individual star. The Vlasov equation can also be written as $(dF/dt)_{\text{orb.}} = 0$, meaning that the total time derivative of the distribution function, evaluated along the orbit of a single star moving in the gravitational potential Φ, vanishes. It has to be stressed that the non-collisional dynamics governed by the Vlasov-Poisson system provides only an approximate description of the behaviour of an N-particle self-gravitating system. However, this approximation is very useful for the study of various stellar systems [8]. A complete (Newtonian) account of the dynamics of a self-gravitating N-particle system is provided by the full set of N coupled (Newtonian) equations of motion of the N-particles. In general, this approach to the dynamics of self-gravitating systems is investigated via numerical N-body simulations (see, for instance, [7]).

The Vlasov equation (1) for self gravitating systems, with the potential Φ given by (2), is nonlinear, since the gravitational potential Φ is given, in a self-consistent way, in terms of the distribution function F. An important feature of the Vlasov-Poisson system is that, given a functional of the form

$$C[F] = \int g(F) d^3\mathbf{x} d^3\mathbf{v}, \tag{3}$$

the solutions of the Vlasov equation (1) verify $dC/dt = 0$. The total energy of the system, given by

$$E = \frac{m}{2} \int \mathbf{v}^2 F(\mathbf{x}, \mathbf{v}) d^3\mathbf{x}\, d^3\mathbf{v} - \frac{Gm^2}{2} \int \frac{F(\mathbf{x}, \mathbf{v}) F(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}\, d^3\mathbf{v}\, d^3\mathbf{x}'\, d^3\mathbf{v}', \tag{4}$$

is, of course, preserved under the evolution governed by equations (1-2).

When stellar systems relax to an equilibrium (or to a meta-equilibrium) state, one expects them to "forget" all information about their initial conditions with the exception of the conserved quantities. Consequently, if one tries to infer by recourse to a maximum entropy prescription the final relaxed state, it is reasonable to use the conserved quantities as constraints. The natural constraints for spherical stellar systems are the total mass and energy of the system. However, if one tries to maximize the standard Boltzmann-Gibbs entropy of the system under the constraints imposed by the conservation of total mass and energy, one obtains the isothermal sphere distribution, which has *infinite* mass and energy [8].

In [2], it was shown that the extremalization of the non extensive q-entropy under the same constraints leads to the stellar polytropic sphere distributions which, for a certain range of the q parameter, are endowed with *finite* mass and energy, as physically expected. This constituted the first clue suggesting that the generalized thermostatistical formalism based on S_q is relevant for the study of systems exhibiting non extensive thermodynamical properties due to long range interactions.

Stellar polytropic sphere distributions are of the form

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}) = f(\tilde{\epsilon}) &= A(\Phi_0 - \tilde{\epsilon})^{n-3/2} \qquad \tilde{\epsilon} \leq \Phi_0 \\ &= 0 \qquad \tilde{\epsilon} > \Phi_0, \end{aligned} \tag{5}$$

where

$$\tilde{\epsilon} = \frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x}), \tag{6}$$

is the total energy (per unit mass) of an individual star, and A, Φ_0, and n (usually called the *polytropic index*) are constants. The quantity $f(\mathbf{x},\mathbf{v})d^3x d^3v$ denotes the number of stars with position and velocity vectors respectively within the elements d^3x and d^3v in position and velocity spaces. The polytropic distribution (5) exhibits, after an appropriate identification of the relevant parameters, the q-MaxEnt form. The entropic parameter q can be related to index n (see Figure 1) by identifying n-3/2 with $q/(1-q)$, obtaining [5]

$$\frac{1}{1-q} = n - \frac{1}{2}. \tag{7}$$

The limit $n \to \infty$ (hence $q = 1$) recovers the isothermal sphere case; $n = 5$ corresponds to the so called Schuster sphere; for $n < 5$ (hence $q < 7/9$), finiteness for mass and energy naturally emerges. The cut-off in the polytropic distribution (5) is an example of what is known, within the field of non extensive thermostatistics, as "Tsallis cut-off prescription", which affects the q-maximum entropy distributions when $q < 1$. In the case of stellar polytropic distributions this cut-off arises naturally, and has a clear physical meaning. The cut-off corresponds, for each value of the radial coordinate r, to the corresponding gravitational escape velocity [8].

Polytropic distributions constitute the simplest, physically plausible models for self-gravitating stellar systems [8]. Alas, these models do not provide an accurate description of the observational data associated with real galaxies. In spite of this, the connection between the S_q entropy and stellar polytropes is of considerable interest. Besides the notable fact that, for a special range of values of q, non-extensive thermostatistics leads to finite stellar systems, the established connection between the S_q entropy and stellar polytropic distributions is interesting for other reasons. Polytropic distributions arise in a very natural way from the theoretical study of self-gravitating systems. The investigation of their properties has constantly interested theoretical astrophysicists during the last one hundred years [8]. Polytropic distributions are still the focus of an intense research activity [3-7], and the study of their basic properties constitutes a part of the standard syllabus of astrophysics students. Now, *polytropic distributions happen to exhibit the form of q-MaxEnt distributions, that is, they constitute distribution functions in the* ($\mathbf{x}$,$\mathbf{v}$) *space that maximize the entropic functional S_q under the natural constraints imposed by the conservation of mass and energy [2].* It is important to remember that the original path leading to the S_q entropic form was not motivated by self-gravitating systems, nor was it motivated by any other *specific* system. The entropic form S_q was proposed by recourse to very general arguments dealing with the consideration of (i) entropic forms incorporating power law structures (inspired on multifractals) and reducing to the standard logarithmic measure in an appropriate limit and (ii) the basic properties that a functional of the probabilities should have in order to represent a physically sensible entropic measure [1]. The q-entropy is a quite natural and, to some extent, unique generalization of the Boltzmann-Gibbs-Shannon measure. Taking this into account, it is remarkable that the extremalization of the q-measure leads to a family of distribution functions of considerable importance in theoretical astrophysics. In a sense, astrophysicists, when studying Newtonian self-gravitating systems, have been dealing with q-MaxEnt distribution functions for over a hundred years without being aware of it. The link between the Tsallis non extensive q-entropy and stellar polytropic distributions consitutes a clue (among several others) indicating that the entropic measure S_q is not just an ad hoc mathematical construction.

As already mentioned, many interesting contributions have been made in recent years in connection with the application of Tsallis' thermostatistics to self gravitating systems in general, and to the stellar polytropes in particular [3-7]. Sau Fa and Lenzi have obtained the exact equation of state for a two-dimensional self gravitating N-particule system within Tsallis thermostatistics [3]. An interesting analysis of the Jean's gravitational instability of a self-gravitational system characterized by a q-gaussian velocity distribution was done by Lima, Silva, and Santos [4]. A detailed study of the gravothermal catastrophe of self-gravitating systems in conection with Tsallis' q-entropy was performed by Taruya and Sakagami (see [5] and references therein). It has been recently pointed out by Chavanis and Sire [6], that the criterion for nonlinear dynamical stability for spherical stellar systems governed by the Vlasov-Poisson equations resembles a criterion of thermodynamical stability. On the basis of this, it is possible to develop a thermodynamical analogy to study the nonlinear dynamical stability of spherical galaxies. Within this approach, the concepts of entropy and temperature would be essentially effective. In [6], the Tsallis functional is interpreted as a useful H-function connecting continuously stellar polytropes and isothermal stellar systems.

On the basis of long term, N-body simulations of self gravitating systems, Taruya and Sakagami have obtained important numerical evidence for a connection between the physics of self-gravitating systems and the Tsallis formalism [7]. Taruya and Sakagami have shown that the evolution of a stellar system confined within an adiabatic wall (starting with initial conditions away from the Boltzmann-Gibbs distribution, and before the system enters the gravothermally unstable regime and undergoes core collapse) can be fitted remarkably well by a sequence of polytropic distributions (that is, Tsallis' q-maxent distributions) with an evolving polytropic index (i.e., evolving q-parameter). Taruya and Sakagami found that the alluded sequence of polytropic distributions provides a good description both of the density profile, and of the single-particle energy distribution of the transient states of the system. Even more interesting, they also found numerical evidence that the same kind of behaviour occurs if the outer boundary is removed. This suggest that the polytropic distributions may play an important role as quasi-attractors, or quasi-equilibrium states of an evolving self-gravitating system [7].

Summing up, we have seen that the connection between Tsallis entropy and self-gravitating systems has been an active research field in recent years, generating a considerable amount of new results. For sure, there are still several open questions to be addressed. For instance, is there any role to be played here by the super-statistics formalism (see the article by Beck, Cohen, and Rizzo in this Issue)? Another interesting question, in our opinion, concerns the possible relationship between the physics of systems with long range interaction, and the recently advanced proposal by Tsallis, Gell-Mann and Sato [9], that special occupancies of phase space may make the S_q entropy additive.

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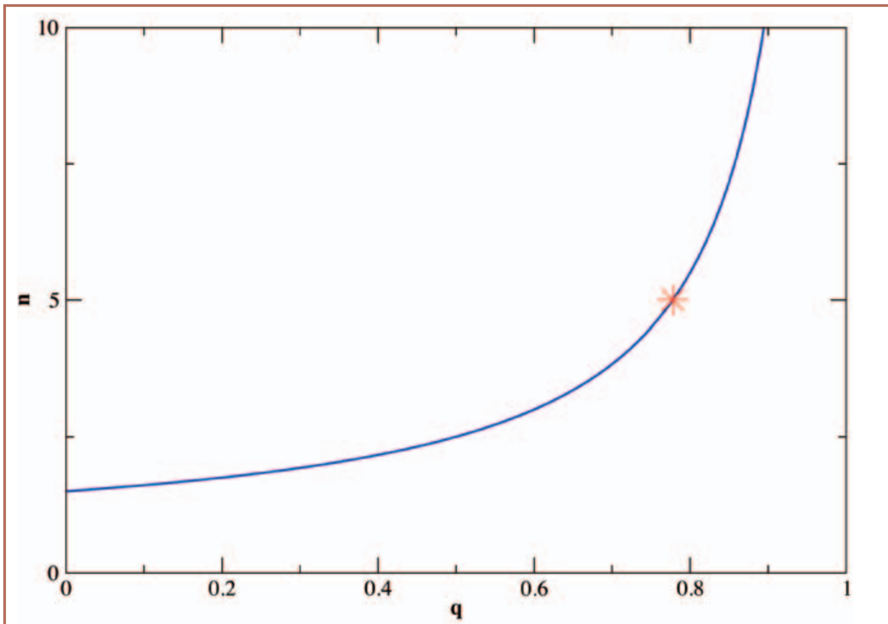

◄ **Fig. 1:** Stellar polytropes are simple models for stellar systems exhibiting a particle density in position-velocity space that is a power-like function of the energy. The conco-mitant exponent is n-3/2, where n is called the polytropic index. This polytropic distribution can be obtained by optimizing the system's q-entropy under the constraints imposed by the system's total mass and energy. The polytropic index n is here plotted as a function of the Tsallis' entropic paramenter q. The red star corresponds to the so called Schuster sphere, with $n = 5$ and $q = 7/9$. For $n < 5$ ($q < 7/9$) the polytropic distributions have finite mass and energy. All the depicted quantities are dimensionless.