mobilities (because of their differences in viscosity or density) in systems such as a shallow layer or a porous medium, when the fluid with highest mobility is forced through the other fluid. As soon the system responds non-linearly to the driving force, enhanced internal fluctuations (such as concentration fluctuations) are produced characterizing the early stage of the fingering process. If the fluids are miscible, the mixing zone at the interface between the two fluids grows as the fluid with high mobility displaces the other fluid, and there is a dynamical transition where the exponent of the growth of the mixing length of the interfacial zone, $L_{mix} \propto t^{\mu}$, changes from $\mu = 1/2$ (the value typical of a diffusive process) to a larger value. In the diffusive regime (before any fingering pattern becomes visible), the flow produces local concentration gradients which induce mobility fluctuations thereby triggering vorticity fluctuations. The concentration field in Fig.1 shows that a "landscape" of alternating hills and wells has developed. In each "blob", the concentration field exhibits a two-dimensional q-Gaussian profile as illustrated in the upper panel of Fig.2 obtained by a section plane cut through the extrema in Fig.1. Such q-Gaussians are precisely solutions to the generalized diffusion equation. Now the remarkable fact is that the distribution that follows from a q-exponential profile has a power law behavior. In two dimensions and for a q-Gaussian – as for the case of the concentration fluctuations c in the fingering pre-transitional regime – the distribution is simply $P(c) \propto c^{-q}$, as illustrated in Fig.2.

The example presented here is representative of a generic class of driven nonequilibrium systems where q-exponentials and power law distributions are the signature of long-range interactions and whose dynamical behavior is governed by non-linear equations, such as the generalized equation described above.

What has been shown is that during the onset of fingering, one can identify precursors which exhibit statistical features typical of nonextensive statistics. Then the question arises as to whether there is a possible physical interpretation of the origin of *nonextensivity?* The driving force produces a spatial sequence of alternating structures, which, if they were independent, would exhibit an ordinary Gaussian profile originating from local diffusion centers (δ-functions), and would be described by a classical advection-diffusion formulation. However, when growing, these structures develop into overlapping Gaussian blobs, and what the analysis shows is that by renormalizing the overlapping Gaussians, they are recast into a *sum* of scale invariant independent q-Gaussians. Similar statistical properties have been obtained in other nonequilibrium systems which are discussed in companion articles in the present issue.

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Relaxation and aging in a long-range interacting system

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Complexity refers to the quality that certain systems possess of being intricate and hardly predictable. Ranging from the turbulent flows that form our atmosphere to the human languages, our life has plenty of examples of natural complex behavior. Statistical Mechanics, the area of physics that deals with the problem of explaining the macroscopic world from the dynamics of its components, faces nowadays the challenge of applying the standard reductionist program to all these fascinating systems.

Even when the mathematical equations for describing its time evolution may be only a few, a complex system is composed of a huge number of interacting constituents. These constituents, usually very simple ones, interact giving rise to the emergence of an unexpected collective phenomenology, where cause and effect become subtle and where the long time behaviour is no longer obvious. For succeeding in the plan of explaining complex behavior from first principles, physicists have been looking for simplified models, mathematically tractable and able to catch the essence of complexity.

Suppose you have such a complete knowledge on the microscopic details of certain system that you can write down its Hamiltonian. Now the natural question that arises is the following one: which are the mechanical conditions the system must fulfill in order to guarantee that the statistical mechanics calculations would predict, with an adequate degree of accuracy, the time averaged quantities obtained from a laboratory experiment. And when trying to answer such an apparently simple question, one discovers that even the simplest systems can give place to very intricate behaviour.

Perhaps the simplest Hamiltonian model of interacting particles one can image is the so called *Hamiltonian Mean Field* (HMF) model. Unlike most of the models we are used to deal with when modeling complexity, in this case, not only the dynamical variables but also the interactions among them are extremely simple, lacking any trait of randomness or frustration. The system consists of a set of N interacting particles or *rotators* of unitary mass, each one confined to move around its own unitary circle [1]. Each particle is then mechanically described by an angle θ_i and the corresponding conjugate momentum p_i. The dynamics of the system is ruled by the following Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2N}\sum_{i,j}\Big[1 - \cos(\theta_i - \theta_j)\Big] \equiv K + V. \quad (1)$$

The first term is the kinetic energy associated with the motion of the particles, while the second one corresponds to the interaction potential (the summation running over all different pairs of particles).

There are a few features of the model that are worth mentioning here. In the first place, it represents a fully connected system, in which each particle interacts with all the others, independently of the distance between them. As it is well known, this unrealistic approach drastically simplifies the mathematical treatment of the thermostatistics of many models, keeping track anyway of its qualitative thermodynamical behavior, at least at high enough dimensionality. Second, the interaction is *ferromagnetic* in nature, in the sense that the potential energy of a pair of interacting particles $[1-\cos(\theta_i-\theta_j)]$ tends to synchronize their movements. Finally, this model can be considered a kinetic version of the *XY* mean field magnetic model, which is without any doubt one of the most studied statistical systems. In fact, we can associate with each particle a two-dimensional local magnetization vector $\vec{m}_i = (\cos\theta_i ; \sin\theta_i)$ and correspondingly a global order parameter:

$$\vec{M} = \frac{1}{N}\sum_i \vec{m}_i \,. \qquad (2)$$

The thermodynamics of this model can be easily solved in the canonical ensemble [1], and this calculation predicts the existence of a continuous phase transition at $T_c = 1/2$ between a *high temperature disordered phase* (characterized by $\vec{M} = \vec{0}$), where rotators uniformly distribute over the circle, and a low temperature ordered phase ($\vec{M} \neq \vec{0}$), where rotators tend to synchronize their movements.

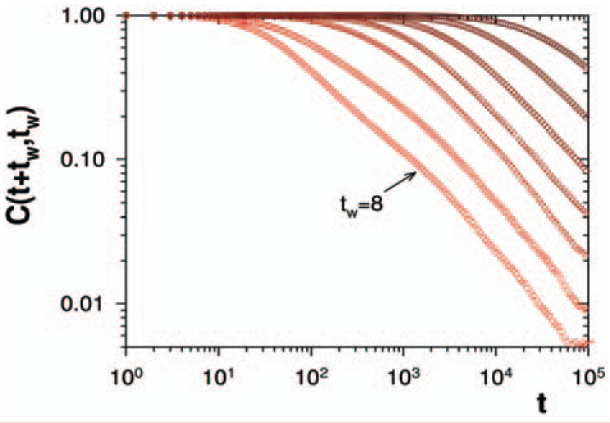
Most of the magnetic models in statistical mechanics do not take into account the kinetic energy. This is mainly because of the well established fact that in any *measure based* statistical theory (microcanonical, canonical or grandcanonical) its contribution to thermodynamical quantities is straightforward (that of a simple ideal gas). However, the inclusion of this term in (1) provides a proper deterministic microscopic dynamics. That is, instead of putting in by hand an external dynamics that would force the system to visit phase space according, for instance, to the usual Boltzmann–Gibbs probability distribution, we can now investigate the true dynamics by simply integrating Newtonian equations

$$\begin{aligned} \dot{\theta}_i &= p_i \\ \dot{p}_i &= M_y \cos\theta_i - M_x \sin\theta_i, \text{ for } 1 \le i \le N \end{aligned} \qquad (3)$$

In doing so, one discovers that, despite its apparent simplicity, this model displays a surprisingly rich variety of complex dynamical behaviour[2].

Let us assume that the system is in thermodynamical equilibrium with a thermal bath at temperature *T*. Then, through the canonical ensemble calculation, we can obtain the mean energy per particle $U/N = \langle H/N \rangle$ by simply assuming, as we learn in any course on Thermostatistics, that the system visits microscopic configurations according to the Boltzmann–Gibbs measure. Alternatively, we may consider a completely isolated system and prepare it initially with a given energy per particle. If we measure the time average of twice the mean kinetic energy per particle $2\bar{K}/N$ along a trajectory, then one would expect this last quantity to coincide with the temperature *T* of the original thermal bath (after suitable transients). This *ergodic* assumption is the master key of the thermostatistics method applied to systems in true equilibrium. But unfortunately complexity seems to occur far away from equilibrium.

There are plenty of physical systems which stay in macroscopic almost stationary states (then, presumably predictable ones) but where the canonical recipe fails. For instance, a window glass or a living cell. Our simple HMF is an excellent prototype for discussing these fascinating questions mainly because, as we will describe in short, the canonical prescription seems to be insuffi- cient to describe its long time behavior. In particular, the HMF is very sensitive to its initial preparation, specially just below the critical point.



▲ **Fig. 1:** Normalized two-time auto-correlation function of the state variable $(\vec{\theta};\vec{p})$ vs. time, for a value of the total energy (subcritical) and for initial conditions that guarantee that the system will get trapped into a quasi stationary trajectory. Data correspond to averages over 200 of such trajectories. The waiting times are $t_w = 8 \times 4^n$, with $n = 0; \ldots; 6$. The dependence of *C* on both times is evident.

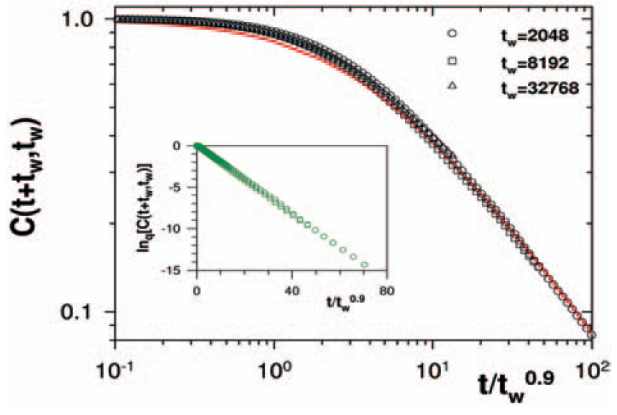
Depending on the initial conditions, the system may become stacked into non-equilibrium long-standing quasi-stationary trajectories. Along these quasi–stationary solutions, whose lifetimes diverge in the limit $N \to \infty$, the time average of any thermodynamical quantity does not coincide with the value predicted by the canonical thermostatistics calculations. The analysis of this complex phenomenology had been the subject of extensive research, including a certain degree of sane controversy [2]. Actually, it presents the kind of drastic slowing down observed in disordered systems, as it happens, for instance, in spin glasses after a sudden quenching into the low temperature phase. Furthermore, the *caloric curve* (the relationship between the internal energy of the system and its temperature), obtained by integrating the equations of motion of the isolated system, strongly disagrees, in the subcritical region, with that expected in the canonical ensemble. Interestingly, this anomalous caloric curve closely resembles the one observed in multifragmentation of clusters of ions or atomic nuclei, where regions with negative specific heat appear. Recently we have shown that this anomalous behavior can be understood in terms of the topology of the potential energy function: the system can not attain true equilibrium because it gets trapped into a sequence of critical points of *V/N* [3], as also verified in many glassy models.

A very simple way of characterizing the relaxation dynamics of a complex system is through the analysis of the two-time auto-correlation function $C(t, t')$, which can exhibit *history-dependent* features, usually referred to as *aging*. For systems that have attained *true* thermodynamical equilibrium, memory effects disappear and only time differences make physical sense. Under these conditions, one expects that $C(t, t') \equiv C(t - t')$. However, for systems exhibiting aging, a much more complex dynamical behavior is observed. (see for instance [4]). Inspired by the strong analogy between the quasistationary trajectories already described and the out of equilibrium states observed in glassy systems, we decided to analyze the behavior of the two-time auto-correlation function of the HMF model [5]. The state of the system in phase space is completely characterized by giving the state vector $\vec{x} \equiv (\vec{\theta}, \vec{p})$. Then, the crude two–time auto-correlation function is

$$C_o(t + t_w, t_w) = \langle \vec{x}\,(t + t_w) \cdot \vec{x}\,(t_w) \rangle, \qquad (4)$$

where $\langle \cdots \rangle$ stands for average over several realizations of the

features



▲ **Fig. 2:** Auto-correlation function vs. scaled time. The data are the same shown in Fig. 1 for the three largest t_w, but suitably scaling the time coordinate makes the data collapse into a single curve. The gray solid line corresponds to a q-exponential fitting. Inset: $\ln_q$-linear representation of the same data, with $q \approx 2.35$. Linearity indicates q-exponential behavior.

dynamics. Afterwards, the auto-correlation function is suitable centered and normalized to remain within the interval [-1, 1].

As we have previously mentioned, because we are dealing with a well defined Hamiltonian system, it is possible to analyze its proper microscopic dynamics. But macroscopic systems always involve such a huge number of interacting particles that the analytical integration of the equations of motion of all the constituents is out of possibility. Then we must be able to integrate their equations of motion with the help of computers (that is what we normally call a numerical simulation). Fortunately, for the system we are interested in, the numerical integration of the coupled equations of motion is a very simple task, which can be carried out even on a modern personal computer, due to its mean-field character. In the cases presented here, the system was always prepared in a "water bag" initial condition, that is, all the angles were set to zero while the momenta were randomly chosen from a uniform distribution with zero mean and such that the system has total energy U. Since for these initial configurations, the total energy is purely kinetic, we emulated the most drastic cooling down compatible with the chosen fixed energy.

In Fig. 1 we plot the normalized two-time autocorrelation function $C(t + t_w, t_w)$, for $U/N = 0.69$, $N = 1000$ and different waiting times (increasing from bottom to top). The value of the specific energy, together with the water bag initial condition, guarantees that, typically, the system will get trapped into a quasi-stationary trajectory as desired. In particular, for the values of U and N chosen, the discrepancy between canonical prediction and microcanonical simulations is the most pronounced one. We clearly note history dependence: for a given fixed t_w, the system remains in a quasi–equilibrium regime with temporal translational invariance up to a time of order t_w. Thereafter, the auto-correlation function presents a slow algebraic decay and a strong dependence on both times. Furthermore, the longer the waiting time t_w, the slower the decay of the correlation.

It is a well established fact that, despite its verified ubiquity in nature, a careful analysis of the aging phenomenology can give valuable information about the microscopic mechanisms involved in the slowing down of the dynamics. In particular, since a general microscopic theory for aging is still lacking, scaling properties can offer a qualitative description of the microscopic phenomenology. Fortunately, a large body of evidence suggests the existence of only a few *dynamical universality classes* associated with the out-of-equilibrium relaxation of a model, as occurs, for instance, in coarsening dynamics or critical phenomena, from which one can extract, by analogy, valuable conclusions. Following these ideas we have looked for the functional dependence of $C(t + t_w, t_w)$ on both times, t_w and t, by trying different data collapses. In Fig. 2 we present the best data collapse obtained for the results of the three largest waiting times displayed in Fig. 1. The resulting scaling law shows that:

$$C(t + t_w, t_w) = f\,(t/t_w^{\beta}) \qquad (5)$$

for the whole range of values of t/t_w considered. Note that, for $t \ll t_w$ it holds that $f\,(t/t_w^{\beta}) \sim (t/t_w^{\beta})^{-\lambda}$. Surprisingly, this kind of scaling behavior is not usual in ordered systems, like the one here studied. Instead, this is the same kind of scaling observed in real spin glasses, which are characterized by the existence of high degrees of randomness and frustration [4]. The solid line corresponds to the best fit of the data with the q-exponential function. This function naturally arises within the generalized thermostatistics introduced by Tsallis [6]. One sees that $q = 1 + 1/\lambda$, yielding $q \approx 2.35$. Notice that the q-exponential allows to fit the whole simulated time span, concluding that:

$$C(t + t_w, t_w) \propto \exp_q\,(-t/t_w^{\beta}) \qquad (6)$$

This affirmation is corroborated by the plot in the inset of Fig. 2, where the same data of the main figure are represented as $\ln_q[C(t + t_w, t_w)]$ vs. t/t_w^{β}, yielding an almost perfect linear behavior. It is worth mentioning that similar fittings were obtained for other system sizes. Later on, these simulations were remade by Pluchino, Rapisarda and Latora [7] who verified that the sub-aging regime observed ($\beta < 1$) is mainly due to the contribution of the momenta. Instead, the contribution of the angles to the correlation function displays the so called *simple aging* regime $C(t, t') \sim t/t_w$, which can be easily understood in terms of the angular coordinates.

Finally, in Fig. 3, we present a further connection between dynamical anomalies and generalized Tsallis thermostatistics. This plot, obtained from [8], presents the probability distribution function (PDF) of the angular position, for a system of $N = 1000$ rotators, initially prepared in the same initial conditions used for analyzing aging, and at different times. It has been reported, some years ago, that, in the quasi-stationary regime, the angles evolve super-diffusively [9] (see also inset of Fig. 3, where the squared deviation is plotted as a function of time). Additional information is obtained looking not only at the squared deviation but at the whole distribution of angles. Curiously, after the meta-equilibrium regime settles, the PDFs can be well described by q-Gaussian shapes. The value of parameter q increases with time reaching a steady value $q \approx 1.5$[8]. The ultimate relationship between this value of q and that obtained previously from the aging curves remains an open question that deserves further analysis [10].

In conclusion, we have discussed two relaxational features (aging and spreading of angles) of the HMF, a paradigm of long-range couplings. Along the quasistationary solutions, both features present traits of generalized exponential behavior. These traits may be a reflection of the complex structure of the phase space regions where quasi-stationary states live. If that were the case, then we would expect that the new generalized thermostatistics introduced by C. Tsallis in 1988 [6], inspired in multifractal geometries, could offer a *novel measure-based* theory for predicting the mean values of a physical system when confined in quasi–stationary long living states. Despite the proper complexity of this enterprise, the manageability of the HMF model invites further and deeper investigations along these lines.

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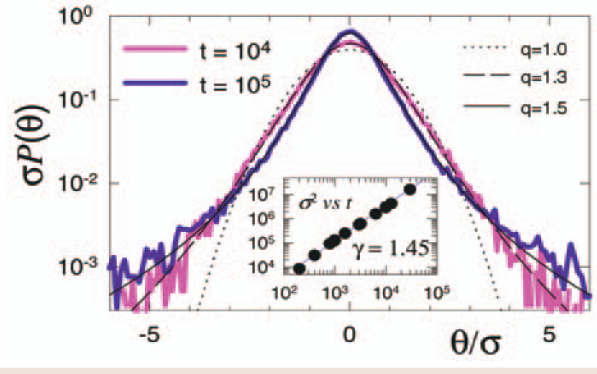

▲ **Fig. 3:** Histogram of normalized angles at different times of the HMF dynamics. Parameters and initial conditions are the same used in previous figures. Notice that at long times, the histogram is of the *q*-Gaussian form. Inset: squared deviation as a function of time. It follows the law $\sigma^2 \sim t^{\gamma}$, with $\gamma > 1$, signaling superdiffusion.

Noise induced phenomena and nonextensivity

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During the last few decades of the 20th century the scientific community has recognized that in many situations (and against everyday intuition) noise or fluctuations can trigger new phenomena or new forms of order, like in noise induced phase transitions, noise induced transport [1], stochastic resonance [2], noise sustained patterns, to name just a few examples. However, in almost all the studies of such noise induced phenomena it was assumed that the noise source had a Gaussian distribution, either white (memoryless) or colored (that is, with "memory"). This was mainly due to the difficulties in handling non Gaussian noises and to the possibility of obtaining some analytical results when working with Gaussian noises. In addition to the intrinsic interest in the study of non Gaussian noises, there has been some experimental evidence, particularly in sensory and biological systems, indicating that at least in some cases the noise source could be non Gaussian.

This article is a brief review on recent studies about some of those noise induced phenomena when submitted to a colored (or time correlated) and non Gaussian noise source. The source of noise used in those works was one generated by a *q*-distribution arising within a nonextensive statistical physics framework [3]. In all the systems and phenomena analyzed, it was found that the system's response was strongly affected by a departure of the noise source from the Gaussian behavior, showing a shift of transition lines, an enhancement and/or marked broadening of the systems response. That is, in most of the cases, the value of the parameter q optimizing the system's response resulted $q \neq 1$ (with $q = 1$ corresponding to a Gaussian distribution). Clearly, this result would be highly relevant for many technological applications, as well as for some situations of biological interest.

Non gaussian noise

In order to introduce the form of the non Gaussian noise to be used, we start considering the following form of a Langevin or stochastic differential equation (that is, a differential equation with random coefficients), with additive noise

$$\dot{x} = f(x, t) + \eta(t), \qquad (1)$$

where $\eta(t)$ is the stochastic or noise source. Usually, it is assumed that such noise source corresponds to a Gaussian distributed variable, having a correlation $C(t - t') = \langle \eta(t)\eta(t') \rangle$. If the noise is "white" (a particular form of Markovian or memoryless process), we have $C(t - t') \sim \delta(t - t')$, while for a typical Ornstein-Uhlenbeck process, we have $C(t - t') \sim \exp[-(t - t')/\tau]$, with τ the "correlation time".

However, motivated by previous work based on a nonextensive thermostatistics distribution [3], it was assumed that the noise $\eta(t)$ was a non Gaussian and non Markovian process (that