subsystems that are (explicitly or tacitly) probabilistically *independent,* then S_{BG} is *extensive* whereas S_q is, for $q \neq 1$, *nonextensive.* This fact led to its current denomination as "nonextensive entropy". However, if what we compose are subsystems that generate a nontrivial (strictly or asymptotically) scale-invariant system (in other words, with important global correlations), then it is generically S_q for a particular value of $q \neq 1$, *and not* S_{BG}, which is *extensive.* Asking whether the entropy of a system is or is not extensive *without indicating the composition law of its elements,* is like asking whether some body is or is not in movement *without indicating the referential with regard to which we are observing the velocity.*

The overall picture which emerges is that Clausius thermodynamical entropy is a concept which can accomodate with more than one connection with the set of probabilities of the microscopic states. S_{BG} is of course one such possibility, S_q is another one, and it seems plausible that there might be others. The specific one to be used appears to be univocally determined by the microscopic dynamics of the system. This point is quite important in practice. If the microscopic dynamics of the system is known, we can in principle determine the corresponding value of q from first principles. As it happens, this precise dynamics is most frequently unknown for many natural systems. In this case, a way out that is currently used is to check the functional forms of various properties associated with the system and then determine the appropriate values of q by fitting. This has been occasionally a point of – understandable but nevertheless mistaken – criticism against nonextensive theory, but it is in fact common practice in the analysis of many physical systems. Consider for instance the determination of the eccentricities of the orbits of the planets. *If* we knew all the initial conditions of all the masses of the planetary system and had access to a colossal computer, we could in principle, by using Newtonian mechanics, determine a priori the eccentricities of the orbits. Since we lack that (gigantic) knowledge and tool, astronomers determine those eccentricities through fitting. More explicitly, astronomers adopt the mathematical form of a Keplerian ellipse as a first approximation, and then determine the radius and eccentricity of the orbit through their observations. Analogously, there are many complex systems for which one may reasonably argue that they belong to the class that is addressed by nonextensive statistical concepts, but whose microscopic (sometimes even mesoscopic) dynamics is inaccessible. For such systems, it appears as a sensible attitude to adopt the mathematical forms that emerge in the theory, e.g. q-exponentials, and then obtain through fitting the corresponding value of q and of similar characteristic quantities.

Coming back to names that are commonly used in the literature, we have seen above that the expression "nonextensive entropy" can be misleading. Not really so the expression "nonextensive statistical mechanics". Indeed, the many-body mechanical systems that are primarily addressed within this theory include long-range interactions, i.e., interactions that are *not* integrable at infinity. Such systems clearly have a total energy which increases quicker than N, where N is the number of its microscopic elements. This is to say a total energy which indeed is nonextensive.

Acknowledgements

The present special issue of Europhysics News is dedicated to a hopefully pedagogical presentation, to the physics community, of the main ideas and results supporting the intensively explored and quickly evolving nonextensive statistical mechanics. The subjects that we have selected, have been chosen in order to provide a general picture of its present status in what concerns both its foundations and applications. It is our pleasure to gratefully acknowledge all invited authors for their enthusiastic participation.

Extensivity and entropy production

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In 1865 Clausius introduced the concept of entropy, S, in the context of classical thermodynamics. This was done, as is well known, without any reference to the microscopic world. The first connection between these two levels of understanding was proposed and initially explored one decade later by Boltzmann and then by Gibbs. One of the properties that appear naturally within the Clausius conception of entropy is the extensivity of S, i.e., its proportionality to the amount of matter involved, which we interpret, in our present microscopic understanding, as being proportional to the number N of elements of the system. The Boltzmann-Gibbs entropy $S_{BG} \equiv -k\sum_{i=1}^{W} p_i \ln p_i$ (discrete version, where W is the total number of microscopic states, with probabilities $\{p_i\}$, and where k is a positive constant, usually taken to be k_B). S_{BG} satisfies the Clausius prescription under certain conditions. For example, if the N elements (or subsystems) of the system are probabilistically independent, i.e., $p_{i_1,i_2,\dots,i_N} = p_{i_1}p_{i_2}\dots p_{i_N}$, we immediately verify that $S_{BG}(N) = NS_{BG}(1)$. If the correlations within the system are close to this ideal situation (e.g., local interactions), extensivity is still verified, in the sense that $S_{BG}(N) \propto N$ in the limit $N \to \infty$. There are however more complex situations (that we illustrate later on) for which S_{BG} is not extensive. The question then arises: *Is it possible, in such complex cases, to have an extensive expression for the entropy in terms of the microscopic probabilities?* The general answer to this question still eludes us. However, for an important class of systems (e.g., asymptotically scale-invariant), one such entropic connection is known, namely

$$S_q = k\frac{1-\sum_{1=1}^{W} p_i^q}{q-1} \quad (q \in \mathcal{R};\ S_1 = S_{BG}). \tag{1}$$

(N = 0)	1	1
(N = 1)	π_{10} π_{11}	1/2 1/2
(N = 2)	π_{20} π_{21} π_{22}	1/3 1/6 1/3
(N = 3)	π_{30} π_{31} π_{32} π_{33}	3/8 5/48 5/48 0
(N = 4)	π_{40} π_{41} π_{42} π_{43} π_{44}	2/5 3/40 1/20 0 0

▲ **Table:** *Left:* Most general set of joint probabilities for N equal and distinguishable binary subsystems for which only the number of states 1 and of states 2 matters, not their ordering. *Right:* Triangle with $\epsilon = 0.5$ and $d = 2$ constructed by modifying the Leibnitz-triangle. In general $q_{sen} = 1-(1/d)$. For $N = 5, 6, \dots$ a full triangle emerges (on the right side) all the elements of which vanish. For any finite N, the Leibnitz rule is not exactly satisfied, but it becomes asymptotically satisfied for $N \to \infty$. See details in [3].

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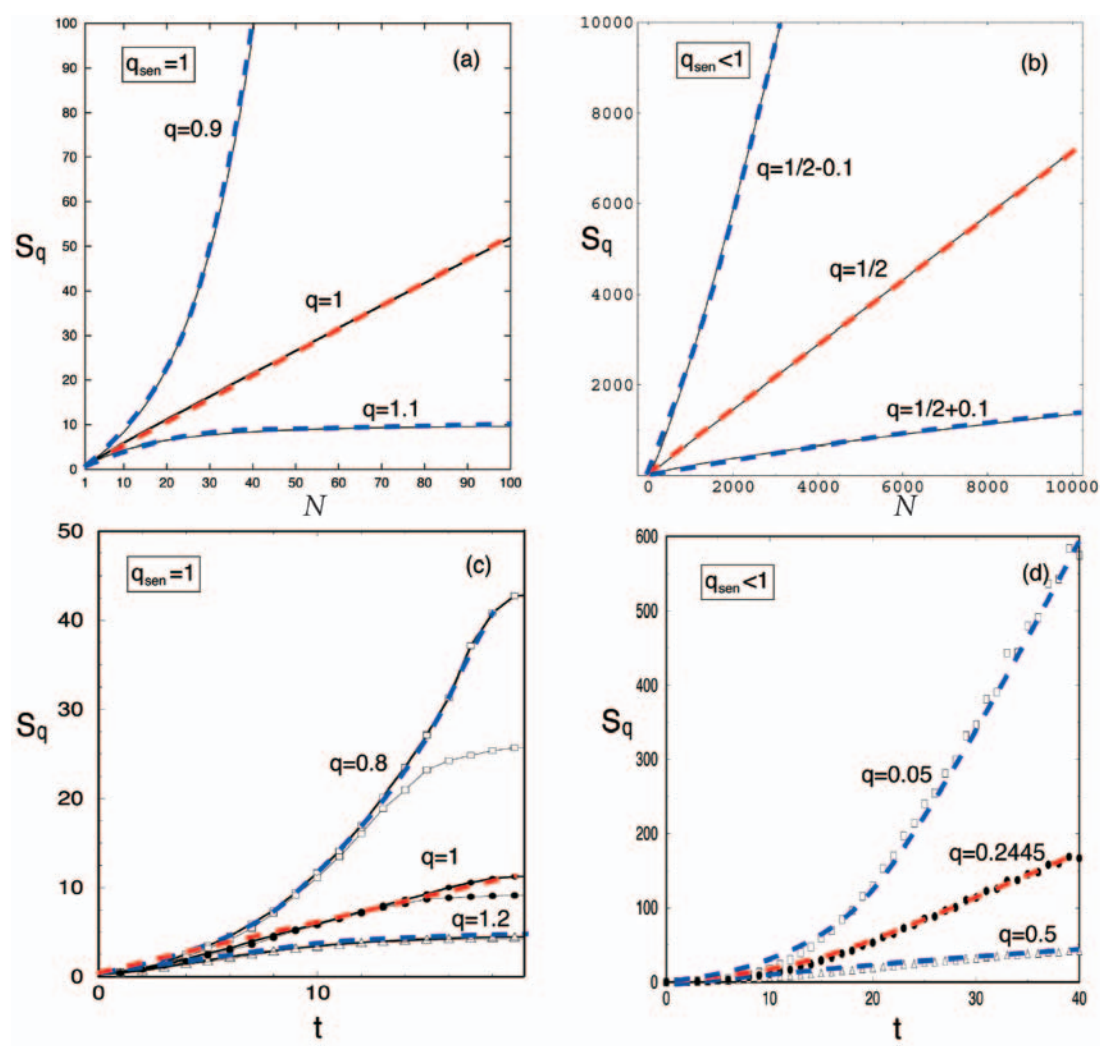
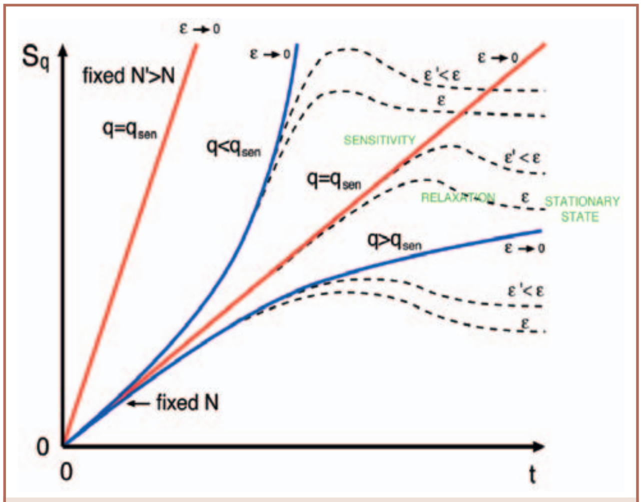


◄ **Fig. 1:** Size-dependence of S_q for probabilistically locally (a) or globally (b) correlated systems (see [3] for details). Time-dependence of S_q for strong (c) and weak (d) chaos (see [4] for details). All the actual data are in black. The coloured dashed lines are visual aids.

This expression was proposed in 1988 [1] as a possible basis for a generalization of Boltzmann- Gibbs statistics currently referred to as *nonextensive statistical mechanics* (see [2] for a set of minireviews). In such a theory the energy is typically nonextensive whether or not the entropy is.

Let us illustrate, for both $q = 1$ and $q \neq 1$, the extensivity of S_q in some examples [3]. Consider a system composed of N identical and distinguishable subsystems (or elements). Let us assume for simplicity that each of those elements corresponds to a probabilistic binary variable which takes values 1 and 2. The joint probabilities of such a system can be represented as in Table I with $\sum_{n=0}^{N} \frac{N!}{(N-n)!\,n!}\pi_{N,n} = 1$ ($\pi_{N,n} \in [0, 1]$; $N = 1, 2, 3, \ldots$; $n = 0, 1, \ldots, N$). Let us impose the scale-invariant constraint $\pi_{N,n} + \pi_{N,n+1} = \pi_{N-1,n}$ ($n = 0, 1, \ldots, N-1$; $N = 2, 3, \ldots$) . Hereafter we refer to this relation as the *Leibnitz rule.* Indeed, it is satisfied by the *Leibnitz triangle:* $(\pi_{10}, \pi_{11}) = (1/2, 1/2)$, $(\pi_{20}, \pi_{21}, \pi_{22}) = (1/3, 1/6, 1/3)$, $(\pi_{30}, \pi_{31}, \pi_{32}, \pi_{33}) = (1/4, 1/12, 1/12, 1/4)$, etc. By inserting these probabilities into expression (1) we can calculate $S_q(N)$ as shown in Fig.1(a). We see that S_q is *extensive only for* $q = 1$. This characterizes a typical *Boltzmannian system.* Let us now consider the probabilities in the table. They have been constructed by starting with Leibnitz triangle, then gradually introducing a zero probability triangle on its "right" side as indicated in Table I [3]. The total measure associated is then redistributed on a strip on the "left side" whose width is d. The distribution is such that $\pi_{N0} \gg \pi_{N1} \gg \ldots$, the discrepancies becoming larger as $N \to \infty$. It can be shown [3] that this system satisfies the Leibnitz rule not strictly but only asymptotically, i.e., for $N \to \infty$. If we now calculate $S_q(N)$ we get the result shown in Fig. 1(b). We see that now S_q is *extensive only for* $q = 1/2$. In fact, for a large class of probability sets, S_q is extensive only for a special value of q, from now on denoted q_{sen} for reasons that will soon become clear (*sen* stands for *sensitivity*). The property $S_q(A + B)/k = [S_q(A)/k] + [S_q(B)/k] + (1 - q)[S_q(A)/k][S_q(B)/k]$, which led to the term "nonextensive entropy", is valid *only* if the subsystems A and B are explicitly or tacitly assumed to be probabilistically independent.

We shall now address a completely different problem, namely that of *entropy production per unit time.* The system now is a specific one, classical and following deterministic nonlinear dynamics. In particular its value of N is fixed. We consider the $D(N)$-dimensional phase space, and denote by W_0 its Lebesgue measure. We then make a partition of it into small cells whose linear size is ϵ. The total number $W(N) \gg 1$ of cells (designated by $i = 1, 2, \ldots, W(N)$) is given by $W(N) \propto W_0/\epsilon^{D(N)}$ with $D(N) \propto N$ ($N \to \infty$). If the phase space is a $D(N)$-dimensional hypercube, then $W(N) = W_0/\epsilon^{D(N)}$. If the system is a classical Hamiltonian one, then $D(N) \sim 2\, d_s N$ ($N \to \infty$), where $d_s \equiv$ *space dimension.*



▲ **Fig. 2:** Schematic time-dependence of S_q for various degrees of fine-graining ϵ. We are disregarding in this scenario the influence of possible averaging over initial conditions that might be necessary or convenient.

We choose one of those cells and in it we randomly pick $M >> 1$ initial conditions. As time t (assumed discrete, i.e., $t = 0, 1, 2, \ldots$) evolves, these M points spread around into $\{M_i(t)\}$ with $\sum_{i=1}^{W(N)} M_i(t) = M$. We can then define a set of probabilities $\{p_i(t)\}$ by $p_i \equiv M_i(t)/M$. With these probabilities we can calculate $S_q(N, t; \epsilon, M)$ for that particular initial cell. Then, depending on our focus, we may or may not average over all or part of the possible initial cells (both situations have been analyzed in the literature). We consider now two different cases, namely *strong chaos* (i.e., the maximal Lyapunov exponent is *positive*), and *weak chaos* (i.e., the maximal Lyapunov exponent vanishes). Both are illustrated in Figs. 1(c) and 1(d) for a very simple system, namely the logistic map $x_{t+1} = 1 - ax_t^2$ with $0 \le a \le 2$, and $-1 \le x_t \le 1$. For infinitely many values of the control parameter a (e.g., $a = 2$), we have strong chaos (this is the case in Fig. 1(c) with $a = 2$). But for other (infinitely many) values of a, we have weak chaos (this is the case in Fig. 1(d) with $a = 1.401155\ldots$). As we can see, it is quite remarkable how strongly similar all four figures 1 are. This suggests the following *conjecture:*

$$\lim_{M\to\infty} S_{q_{sen}}(N,t,\epsilon,M) \sim AN s_{q_{sen}}(t,\epsilon) \;\; (t=0,1,2,...;t>>1;\; N>>1), \quad (2)$$

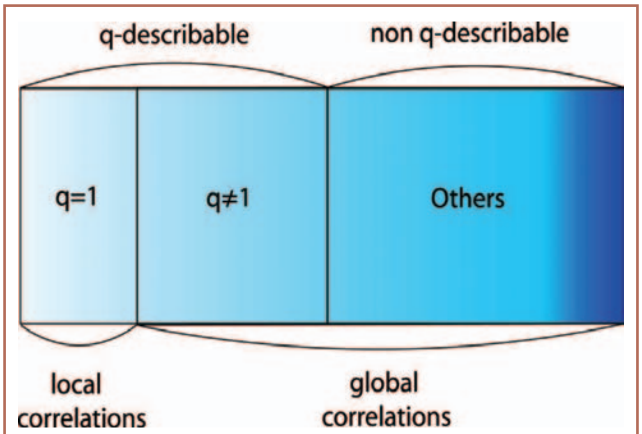
as schematised in Fig. 2. (A is a positive constant.) We emphasize that this conjecture is built to some extent upon observations made on the time-dependence of *low*-dimensional systems, such as the logistic map and similar dissipative maps ([5] and references therein) as well as two-dimensional conservative maps [6]. Whether similar behavior indeed holds for the time-dependence of *high*-dimensional dissipative or Hamiltonian systems with $N >> 1$ obviously remains to be checked.

Conjecture (2) has two consequences. The first of them is that, since by definition of q_{sen} it is $\lim_{\epsilon\to 0} S_{q_{sen}}(t, \epsilon) \sim K_{q_{sen}} t$, we have that $\lim_{\epsilon\to 0} \lim_{M\to\infty} S_{q_{sen}}(N, t; \epsilon, M) \sim AK_{q_{sen}} Nt$. This means, interestingly enough, that N and t play similar roles. The second consequence concerns the case when we have a fine but *finite* graining ϵ, for example that imposed by quantum considerations. Then we typically expect the expressions $\lim_{N\to\infty} \lim_{t\to\infty}$

$$\lim_{M\to\infty} \frac{S_{q_{sen}}(N,t,\epsilon,M)}{Nt} \text{ and } \lim_{t\to\infty} \lim_{N\to\infty} \lim_{M\to\infty} \frac{S_{q_{sen}}(N,t,\epsilon,M)}{Nt}$$

to coincide for typical $q = 1$ systems, and to differ for more complex systems ($q \neq 1$), as might well be the case for long-range-interacting Hamiltonians [7].

Let us illustrate, for a one-dimensional map, an important property associated with Eq. (2). The sensitivity ξ to the initial conditions is defined through $\xi \equiv \lim_{\Delta x(0)\to 0} \Delta x(t)/\Delta x(0)$, where $\Delta x(0)$ is the discrepancy of two initial conditions. For a wide class of one-dimensional systems we have the upper bound $\xi = e_{q_{sen}}^{\lambda_{q_{sen}} t}$, where $e_q^x \equiv [1+(1-q)x]^{1/(1-q)}$ ($e_1^x = e^x$), and $\lambda_{q_{sen}}$ a q-generalised Lyapunov coefficient . The property we referred to is that the entropy production per unit time $K_{q_{sen}}$ satisfies $K_{q_{sen}} = \lambda_{q_{sen}}$ [8]. This generalises, for $q \neq 1$, a relation totally analogous to the Pesin identity, which plays an important role in strongly chaotic systems (i.e., $q_{sen} = 1$). It is clear that K_q is a concept closely related to the so called Kolmogorov-Sinai entropy. They frequently, but not always, coincide. As we have shown, S_q can, for either $q = 1$ or $q \neq 1$, be extensive under suitable conditions and lead to a finite entropy production per unit time. Other important properties are satisfied, such as *concavity* and *Lesche-stability* (or *experimental robustness*) [9]. Moreover, the celebrated uniqueness theorems of Shannon and of Khinchine have also been q-generalised [10,11], and the same has been done with central procedures such as the Darwin-Fowler steepest descent method [12]. In short, a consistent mathematical structure is in place suggesting that the Boltzmann-Gibbs theory can be satisfactorily extended to deal with a variety of complex statistical mechanical systems. Since the first physical application [13] (to stellar polytropes), nonextensive statistical mechanics and its related concepts have made possible applications to very many natural and artificial systems, from turbulence to high energy and condensed matter physics, from astrophysics to geophysics, from economics to biology and computational sciences (e.g., signal and image processing). Recently, connections with scale-invariant networks, quantum information, and a possible q-generalisation of the central limit theorem [14,15] have been advanced as well. In some of these problems, when the precise dynamics is known, the indices q are in principle computable from first principles. In others, when neither the microscopic nor the mesoscopic dynamics is accessible, only a phenomenological approach is possible, and then q is determined through fitting. An interesting determination of this kind was recently carried out in the solar wind as observed by Voyager 1 in the distant heliosphere [16]. Indeed, the q-triplet that had been conjectured was fully determined for the first time in a physical system. The overall scenario which emerges is indicated in Fig. 3.



▲ **Fig. 3:** Scenario within which nonextensive statistical mechanics is located (see [3] for more details) [18].

There is a plethora of open problems, as can be easily guessed. Both at the level of the foundations (e.g., the dynamical origin [17]) and at that of specific applications. The fact that some basic questions are not yet fully understood even for Boltzmann-Gibbs statistics does not make the task easy. As an illustration of an important open problem let us mention *long-range-interacting* Hamiltonians. Although many favorable indications are available in the literature, it is still unknown, strictly speaking, if and how the present theory is applicable, and what is the value of q as a function of the range of the forces and of the space dimension. Solutions of problems such as this one are obviously very welcome. Let us finally mention that related or even more general approaches than the present one are already available in the literature. Such is the case of the Beck-Cohen superstatistics and the Kaniadakis statistics, that have already shown interesting specific applications.

Acknowledgement

Y. Sato has benefited from a Postdoctoral Fellowship at the Santa Fe Institute.
C.Tsallis has benefited from sponsoring by SI International and AFRL.
M. Gell-Man was supported by the C.O.U.Q. Foundation and by Insight Venture Management.

References

[1] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).

[2] M. Gell-Mann and C. Tsallis, eds., *Nonextensive Entropy - Interdisciplinary Applications,* (Oxford University Press, New York, 2004). For a regularly updated bibliography see http://tsallis.cat.cbpf.br/biblio.htm.

[3] C. Tsallis, M. Gell-Mann and Y. Sato, *Proc. Natl. Acad. Sc. USA* **102**, 15377 (2005).

[4] V. Latora, M. Baranger, A. Rapisarda and C. Tsallis, *Phys. Lett. A* **273**, 97 (2000).

[5] E. Mayoral and A. Robledo, *Phys. Rev. E* **72**, 026209 (2005).

[6] G. Casati, C. Tsallis and F. Baldovin, *Europhys. Lett.* **72**, 355 (2005).

[7] V. Latora, A. Rapisarda and C. Tsallis, *Phys. Rev. E* **64**, 056134 (2001).

[8] F. Baldovin and A. Robledo, *Phys. Rev. E* **69**, 045202(R) (2004).

[9] S. Abe, *Phys. Rev. E* **66**, 046134 (2002).

[10] R.J.V. dos Santos, *J. Math. Phys.* **38**, 4104 (1997).

[11] S. Abe, *Phys. Lett. A* **271**, 74 (2000).

[12] S. Abe and A.K. Rajagopal, *J. Phys. A* **33**, 8733 (2000).

[13] A.R. Plastino and A. Plastino, *Phys. Lett. A* **174**, 384 (1993).

[14] C. Tsallis, *Milan J. Math.* **73**, 145 (2005).

[15] L.G. Moyano, C. Tsallis and M. Gell-Mann, cond-mat/0509229 (2005).

[16] L.F. Burlaga and A.F. Vinas, *Physica A* **356**, 375 (2005).

[17] E.G.D. Cohen, *Pramana* **64**, 635 (2005).

[18] The case of standard critical phenomena deserves a comment. The *BG* theory explains, as is well known, a variety of properties as close to the critical point as we want. If we want, however, to describe certain discontinuities that occur *precisely at the critical point* (e.g. some fractal dimensions connected with the $d_s = 3$ Ising and Heisenberg ferromagnets at T_c), we need a different theoretical approach.

Atmospheric turbulence and superstatistics

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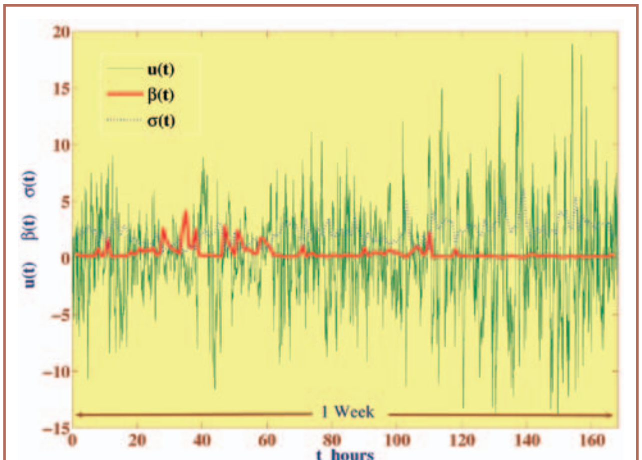
In equilibrium statistical mechanics, the inverse temperature β is a constant system parameter – but many nonequilibrium systems actually exhibit spatial or temporal temperature fluctuations on a rather large scale. Think, for example, of the weather: It is unlikely that the temperature in London, New York, and Firenze is the same at the same time. There are spatio-temporal temperature fluctuations on a rather large scale, though locally equilibrium statistical mechanics with a given fixed temperature is certainly valid. A traveller who frequently travels between the three cities sees a 'mixture' of canonical ensembles corresponding to different local temperatures. Such type of macroscopic inhomogenities of an intensive parameter occur not only for the weather but for many other driven nonequilibrium systems as well. There are often certain regions where some system parameter has a rather constant value, which then differs completely from that in another spatial region. In general the fluctuating parameter need not be the inverse temperature but can be any relevant system parameter. In turbulent flows, for example, a very relevant system parameter is the local energy dissipation rate ϵ, which, according to Kolmogorov's theory of 1962 [1], exhibits spatio-temporal fluctuations on all kinds of scales. Nonequilibrium phenomena with macroscopic inhomogenities of an intensive parameter can often be effectively described by a concept recently introduced as 'superstatistics' [2]. This concept is quite general and has been successfully applied to a variety of systems, such as hydrodynamic turbulence, atmospheric



▲ **Fig. 1:** Time series of a temporal wind velocity difference $u(t)$ ($\delta = 60$ min) recorded by anemometer A every 5 min for one week (green line) and the corresponding parameter $\beta(t)$ (red line), as well as the corresponding standard deviation $\sigma(t)$ (blue dotted line), both for a 1 hour window.

features