Deterministic Diffusion: A Chaotic Phenomenon

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A particular manifestation of chaotic behaviour is the generation of a deterministic "random walk". In the following, some aspects of recent work on this phenomenon are reviewed.

Within the last few years the physics community has rapidly taken notice of the fact that deterministic systems can exhibit erratic, seemingly irregular motions. This phenomenon is known as chaotic behaviour and must be distinguished from erratic motions that are generated by random external forces. The latter is the case *e.g.* for Brownian motion, where the erratic motion of a heavy particle (pollen) is due to random collisions with the molecules of a surrounding liquid. Chaotic behaviour, however, refers to erratic motions that are generated *within* a dynamical system in the *absence* of external randomness. A general prerequisite for its occurrence is the existence of non-linearities in the equations of motion. A number of popular reviews on certain aspects of chaos are already available, the most recent ones being by Kadanoff and by Ford [1]).

Chaotic behaviour may have different features, which may be distinguished by certain physical quantities or mathematical criteria. The erratic motion may be merely nonperiodic, it may be ergodic or mixing. It may depend sensitively on the initial conditions, in which case a *practical* unpredictability is implied since initial conditions usually cannot be measured with ultimate precision. Another criterion may be behaviour "as random as a coin toss" [1]). Recently, simple deterministic systems have been found which may be classified to be "as random as a random walk" [2]). Although similar to Brownian motion, this random walk is not generated by random external forces but stems from the intrinsic nonlinear dynamics of the systems. It is a deterministic random walk, i.e. fixing the initial conditions determines all the future evolution.

Driven Josephson Junction, Particle in a Periodic Potential

As a physical example, consider a particle in a periodic potential, Fig. 1 (as *e.g.* a defect ion in a crystal lattice). This particle is damped and driven by a periodic force; it obeys the equation of motion:

$$\ddot{x} = -\gamma\dot{x} - \omega_0^2 \sin x + F\cos\omega t \quad (1)$$

where x is the position of the particle, γ is the damping coefficient, ω_0 is the frequency of small vibrations at the bottom of the potential wells, F is the amplitude of the driving force and ω its frequency. Driven Josephson junctions satisfy the same equation of motion. In that case the variable x represents the phase difference ϕ between two superconductors weakly coupled through a small contact or a thin oxide layer.

Numerical simulations of this equation revealed an entire zoo of possible dynamical behaviour depending on the choice of parameters [3]). Besides period-doubling and intermittency, a diffusive motion was observed as displayed at the bottom of Fig. 1. In the course of time the particle moves randomly in either direction crossing potential barriers like a diffusing ion in a crystal. This diffusive motion must be of a deterministic nature since random external forces are absent. This is only one example and it changes in character when the parameters are varied. In driven Josephson junctions the diffusion can easily be detected by a measurement of the voltage power spectrum as will be discussed later.

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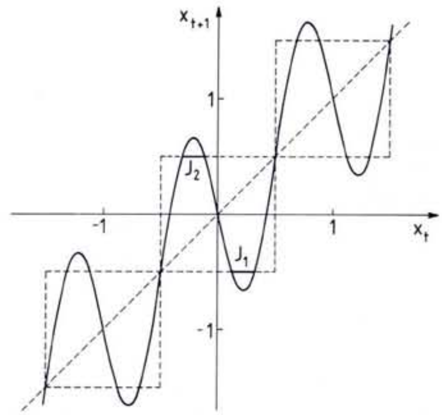


Fig. 2 — One-dimensional mapping Eq. (2), by which a value x_t at time t generates its successor x_{t+1}. When x_t falls within an overlap region J_1 or J_2, x_{t+1} is transfered to a neighbouring cell.

One-Dimensional Mappings

The above differential equation is too unwieldy for an analytical investigation. Let us consider a simpler model [2]), the significance of which will become clear later

$$x_{t+1} = x_t - \mu\sin(2\pi x_t) \quad (2)$$

Here $t = 0,1,2,...$ is a discrete time, x_t a dynamical variable and μ a control parameter corresponding to an externally controllable quantity in an experiment (e.g. driving frequency). This equation is illustrated in Fig. 2. Given an initial value x_0, iteration of Eq. (2) deterministically generates a time series (or orbit) $x_0, x_1, x_2, ...$. For large enough values of the parameter μ the series can perform a random walk as exhibited by Fig. 3. The mean-square displacements diverge linearly in time reflecting a diffusion along the x-axis. For $\mu < \mu_c = 0.733$, the mean-square displacements remain bounded and diffusion does not occur.

The nature of this diffusion can be understood from Fig. 2. The periodicity of the sinusoidal term in Eq. (2) defines unit cells along the x_t-axis and the x_{t+1}-axis as shown by the dashed boxes in Fig. 2. In the regions J_1 and J_2 (transfer regions), x_{t+1} laps over into neighbouring cells. If this were not the case, x_{t+1} would remain in the same unit cell as x_t and diffusion would never occur. When x_t hits a transfer region, x_{t+1} will be in a neighbouring cell. If the transfer regions are small and the chaotic motion within the cell is ergodic, the orbit x_t wanders about for a long time within a cell before it hits a transfer region and a jump into a neighbouring cell occurs. This waiting time can become so large that successive jumps are uncorrelated on the average. The jump rate and thus the diffusion coefficient are expected to be proportional to the width of the transfer regions.

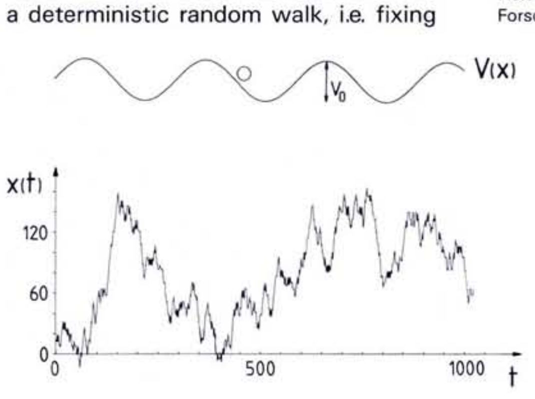


Fig. 1 — Top: particle moving in a periodic potential, as is the case in crystal lattices. It is assumed to be subject to periodic driving and damping. Bottom: time record of the position x(t) demonstrating a diffusive motion over many potential wells (period 2π). Other dynamical behaviour can be observed depending on the choice of parameters. (After A. Reithmayer, diploma thesis, Universität Regensburg, 1981).

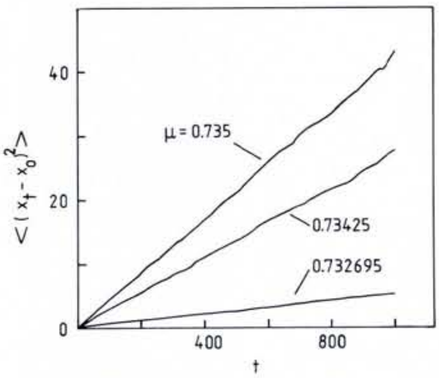


Fig. 3 — Mean-square displacements as a function of time, demonstrating diffusive motion. The slopes of the curves equal twice the diffusion coefficient.

Based on these properties a theory has been constructed which may also be applied to driven Josephson junctions and similar systems. The dynamics of dissipative systems like Eq. (1) mostly reduces to one-dimensional mappings

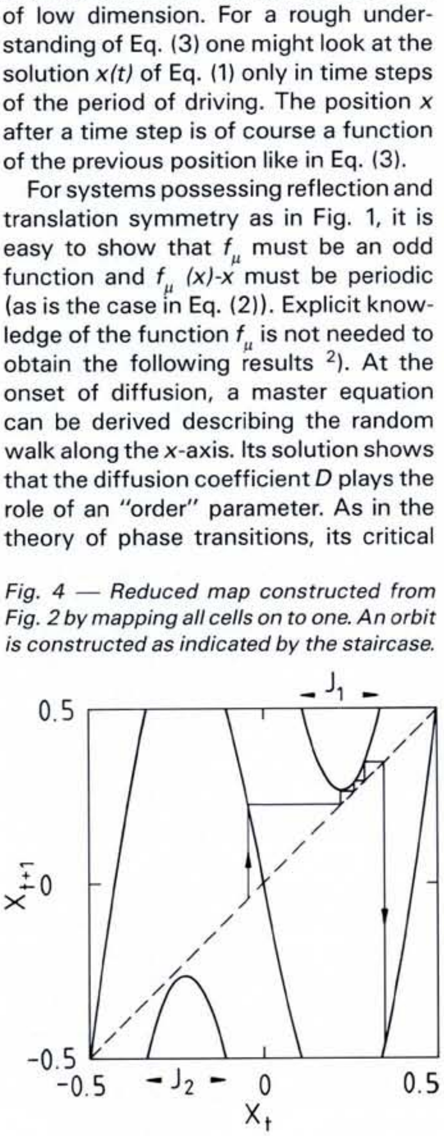
$$x_{t+1} = f_\mu(x_t) \tag{3}$$

where f_μ is a function depending on a parameter μ. The deeper reason for this is the existence of strange attractors [4]) of low dimension. For a rough understanding of Eq. (3) one might look at the solution $x(t)$ of Eq. (1) only in time steps of the period of driving. The position x after a time step is of course a function of the previous position like in Eq. (3).

For systems possessing reflection and translation symmetry as in Fig. 1, it is easy to show that f_μ must be an odd function and $f_\mu(x)$-x must be periodic (as is the case in Eq. (2)). Explicit knowledge of the function f_μ is not needed to obtain the following results [2]). At the onset of diffusion, a master equation can be derived describing the random walk along the x-axis. Its solution shows that the diffusion coefficient D plays the role of an "order" parameter. As in the theory of phase transitions, its critical behaviour as a function of μ-μ_c is described by a power law with a critical exponent. If external noise is added, the dependence on the noise and on μ is given by a universal scaling function.

The theory has now been worked out in some detail [2,5]). Due to translation symmetry it is convenient to consider a reduced map [5]) where all cells are mapped onto a single one. Fig. 2 thereby turns into Fig. 4, where the overlapping parts of the map (in the regions J_1 and J_2) are translated back to the unit cell. One most remember that whenever x_t hits a transfer region J_1 or J_2, a jump to a neighbouring cell is made.

Fig. 4 — Reduced map constructed from Fig. 2 by mapping all cells on to one. An orbit is constructed as indicated by the staircase.



Intermittent Diffusion and Excess Noise

When the control parameter μ is varied, different drifting periodic orbits arise corresponding to a non-zero average drift velocity of the particle in Fig. 1. Before a drifting orbit is reached (via a so-called saddle-node bifurcation) we find some anomalous behaviour as shown in Fig. 5 [6]). The mean-square displacements grow like t^2 up to a crossover time and then follow the standard linear growth. The crossover time may become arbitrarily large. This is accompanied by excess noise at low frequencies; i.e. towards small frequencies the spectrum of velocity fluctuations shows an anomalous increase like ω^{-2}. At zero frequency, the spectrum saturates at a finite value.

The nature of this phenomenon can be easily understood from Fig. 4. An orbit $x_0, x_1, x_2, \ldots$ is generally constructed as indicated by the staircase. In iterating the map, the ordinate x_{t+1} is successively re-used as abscissa x_t. This is done by successive reflections at the dashed diagonal, where $x_{t+1} = x_t$. As is seen, an orbit entering the transfer region J_1 can stay a certain time before being ejected. This implies consecutive jumps to neighbouring cells in every time step during this time. These correlated jumps explain the excess noise and the mean-square displacements observed in Fig. 5. The correlation time becomes arbitrarily long when the map approaches tangency in Fig. 4.

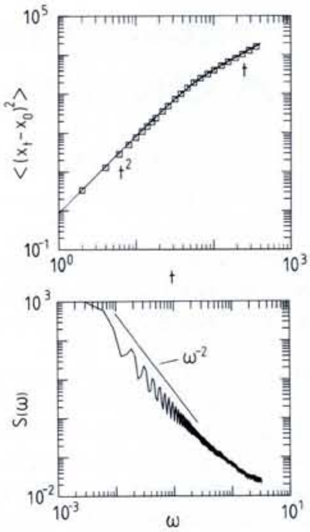


Fig. 5 — Log-log plots of the mean-square displacements (top) and velocity power spectrum (bottom) pertaining to intermittent diffusion. This arises when the map in Fig. 4 approaches tangency. The excess noise in the spectrum falls off like ω^{-2} in a certain frequency regime.

Anomalous Diffusion

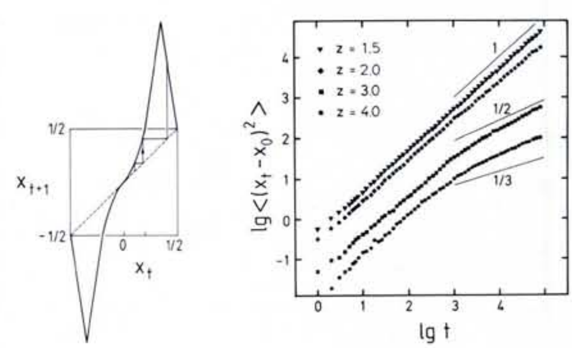
The (Gaussian) diffusion processes most frequently found in nature show a linear growth of mean-square displacements for long times. However, anomalous diffusion has also been found, where linear growth is not obeyed. This has been explained in the past by spatial disorder or fractal structure. It is now demonstrated that anomalous diffusion can also be caused by a specifically chaotic mechanism [6]). Here the nonlinear growth of mean-square displacements persists as $t \to \infty$.

The map in Fig. 2 has fixed points ($x_{t+1} = x_t$) at -1/2, 1/2, 3/2, etc. Such fixed points are also present in the generalizations discussed after *e.g.* Eq. (3). When the control parameters of such a system are varied, the fixed points may become marginally stable (slope of the map = 1); *i.e.* a map may be distorted as shown in Fig. 6, where the origin has been shifted to such a fixed point. In its vicinity the map has the expansion ($x_t \to +0$)

$$x_{t+1} = x_t + a x_t^z \tag{4}$$

where an exponent $z > 1$ has been assumed for greater generality. For the

Fig. 6 — Example of a distorted map (left) where a fixed-point becomes marginally stable (here at the origin). This is associated with an anomalous (i.e. nonlinear) growth of the mean-square displacements (right), which theoretically persists to infinity.



following results, only this limiting behaviour is relevant; away from the origin, the map is largely arbitrary.

The mean-square displacements were determined numerically for this example and are also shown in Fig. 6. They agree with and illustrate our more general theoretical results: For $z<2$ the diffusion is normal. For $z>2$ the mean-square displacements grow anomalously like $t^{1/(z-1)}$. In the limiting case $z=2$ they grow like $t/\ln t$. Note that these are asymptotic results for $t\to\infty$. A crossover to normal diffusion for long times does not occur. The statements are universal, *i.e.* the type of growth depends only on the exponent z and not on details of the map. The origin of this anomalous diffusion can be understood from the left hand side of Fig. 6. An orbit x_0, x_1, x_2, ... indicated by the staircase remains within the cell ($x_t<1/2$) for a certain time then reaches the boundary and a jump is made to a neighbouring cell. The residence time T may become arbitrarily long near the origin. The distribution of residence times has a long-time tail causing the anomalous diffusion. It has been shown by Manneville that maps with a limiting form as in Eq. (4) can generate $1/f$-noise.

Experimental Detection

As mentioned before, prominent examples for deterministic diffusion are a particle in a periodic potential and a Josephson junction driven by periodic forces (Eq. (1)). In many cases their dynamics can be described by maps as studied above. There are four parameters in Eq. (1) that can be varied externally, whereby the corresponding map can be distorted in many ways. In view of the 4-dimensional parameter space it seems plausible that special situations as in Fig. 6 can be realized. Intermittent diffusion as in Fig. 5 has already been observed in numerical simulations of driven Josephson junctions.

The diffusion in driven Josephson junctions can be easily detected in the following way: The diffusing variable x represents the phase difference ϕ, and its velocity $\dot{\phi}$ according to Josephson is proportional to the voltage U across the junction. The velocity power spectrum $S(\omega)$ (Fig. 5) is thus proportional to the voltage power spectrum, which can be measured easily. Diffusion will be reflected in excess noise as in Fig. 5. Moreover the diffusion coefficient can be measured; it is generally related to the velocity power spectrum at zero frequency as in $D=(1/2)S(\omega=0)$.

Very recently we have learned of the experimental observation of such excess noise in the voltage power spectrum of Josephson junctions shunted by a resistance with a self-inductance. In a still unpublished experiment, J. Clarke and R.F. Miracky found the excess noise falling off as ω^{-2}. The result can be explained by intermittent diffusion as treated above.

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Farewell to P.L. Kapitza

EPS is sad to have lost one of its Honorary Members. Peter L. Kapitza died early in April at the age of 85. Awarded the Nobel Prize in physics in 1978 (Europhysics News, 10 (1979) 5, p. 5) he is seen here at the close of the traditional Lindau gathering in July 1982. In his message of condolence to the USSR Academy of Sciences, the President of EPS wrote: "although we say farewell to a great man, his work as a scientist lives on for ever".

(Photo: W.S. Newman)

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